Overview of Particle Beam Optics
Utilized in Matrix, Envelope, and Tracking Codes:
TRACE 3-D, Beamline Simulator (TRANSPORT & TURTLE)

George H. Gillespie
G. H. Gillespie Associates, Inc.
P. O. Box 2961
Del Mar, California 92014, U.S.A.

Presented at
King Abdulaziz City for Science and Technology (KACST)
Riyadh, Saudi Arabia
October 2014
Presentation Outline - Part I

Overview of Particle Beam Optics Utilized in the Matrix, Envelope, and Tracking Codes:
TRACE 3-D, Beamline Simulator (TRANSPORT & TURTLE)

1. Basic Matrix Premise, Coordinates, Linear / Nonlinear Particle Optics, … pp 3-13
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Representations pp 14-30
   ⇒ Break
3. Equations of Motion: Drifts, Quads, Bends - Individual Particle Motion pp 31-54
   ⇒ Break
4. Introduction to the Beam Optics of TRACE 3-D pp 57-74
5. Introduction to the Beam Optics of Beamline Simulator pp 75-81
6. Summary page 82

Part II ⇒ Use the PBO Lab TRAC 3-D Module to work some examples
1. **Basic Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...**

   - Particle optics utilizes a perturbation approach to beam dynamics
   - Motion measured with respect to Reference (or Synchronous) Trajectory
   - Origin of the coordinate system moves along the Reference Trajectory

![Diagram](image)

**Describing Trajectories and Coordinate Systems**

- **Distance** along the Reference Trajectory is denoted here by \( s \)
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont'd)

Reference Trajectory can be thought of as a machine property, often specified in terms of "floor coordinates" (denoted below by subscripts $F$)

Reference Trajectory in a Bending Magnet

At time $t_i$ the reference trajectory enters a bending magnet with initial values of Floor Coordinates $x_{Fi}, y_{Fi}, z_{Fi}$. Inside the bending magnet the reference trajectory follows an orbit in Floor Coordinates given by:

\[
x_F = x_{Fi} - \rho_o \cos(\omega_o[t-t_i])\cos(\phi_i) + \rho_o \sin(\omega_o[t-t_i])\sin(\phi_i)
\]

\[
y_F = y_{Fi}
\]

\[
z_F = z_{Fi} + \rho_o \sin(\omega_o[t-t_i])\cos(\phi_i) - \rho_o \cos(\omega_o[t-t_i])\sin(\phi_i)
\]
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont’d)

• In addition to the "floor coordinates" the Reference Trajectory also has a "Reference Velocity" $v_s$ associated with it.

• A "Reference Particle" (not necessarily any actual particle) moves along the Reference Trajectory at the Reference Velocity.

• The magnitude of the Reference (or Synchronous) Velocity is often denoted $v_s = c\beta_s$, where $c$ is the speed of light and $\beta_s$ is the relativistic speed.

• Similarly one can define a Reference Kinetic Energy, Reference Total Energy, Reference $\gamma_s = (1-\beta_s^2)^{-1/2}$, etc.

• PBO Lab Global Parameters Set Several Initial Reference Trajectory Values

• Some beam optics codes compute the floor coordinates of the Reference Trajectory but many do not.

  TRACE 3-D (and TRANSPORT) compute Reference Kinetic Energy (and/or related parameters) as well as Reference Trajectory length

  TRACE 3-D does not compute floor coordinates

  TRANSPORT does compute floor coordinates

• TRACE 3-D also utilizes a Reference, or Synchronous, Phase $\phi_s$. 
1. **Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont'd)**

So, motion measured with respect to **Reference** (or **Synchronous**) **Trajectory**

![Diagram](image)

**Describing Trajectories and Coordinate Systems**

- **Particle Optics**: describe \([x_b, y_b, z_b]\) in terms of \([x_a, y_a, z_a]\) as function of \(s\)
- **Envelope Optics**: describe **moments** of a distribution \(f_b(x_b, y_b, z_b)\) in terms of **moments** of the distribution \(f_a(x_a, y_a, z_a)\) as function of \(s\)
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont’d)

• A map, $M$, can be used to compute $[x_b, y_b, z_b]$ from $[x_a, y_a, z_a]$. The momentum associated with each will coordinate also be needed, e.g. $[P_x, P_y, P_z]$.

• If we denote the 6-vector $[x_b, P_x, y_b, P_y, z_b, P_z]$ by $[q_i]$ with $i = 1, \ldots, 6$ then $M$ maps $[q_i]$ into $[q_i]$:  
  
  $$[q_i] = M[q_i]$$

• Since all elements of the 6-vectors $[q_i]$ and $[q_i]$ are presumed "small" we should be able to represent the map $M$ by a Taylor series expansion:  

  $$M[q_i] = \Sigma_j R_{ij} q_j + \Sigma_{j\leq k} \Sigma_k T_{ijk} q_j q_k + \Sigma_{j\leq k \leq l} \Sigma_l U_{ijkl} q_j q_k q_l + \ldots$$

• In first-order optics, only the first term is used:  
  
  $$[q_i] = M[q_i] = \Sigma_j R_{ij} q_j$$

  First-order optics $\Rightarrow$ linear optics, described by R-matrix

• In second-order optics, the first 2 terms are used:  
  
  $$[q_i] = M[q_i] = \Sigma_j R_{ij} q_j + \Sigma_{j\leq k} \Sigma_k T_{ijk} q_j q_k$$

  Second- (and higher-)order $\Rightarrow$ nonlinear optics

• Less than or equal sums (e.g. $j\leq k$) avoid double counting (e.g. $T_{ijk}=T_{ikj}$).
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont’d)

Why use this matrix formalism?

• Consider particle starting on-axis: $x_a = 0$ and $y_a = 0$ at Reference Velocity $[v_s]$

• The change in the x-coordinate at the end of a system $[x_b]$ due small initial velocities $[v_{xa}, v_{ya}]$ away from the axis can be written as:

$$x_b = R_{12} [x']_a + R_{14} [y']_a$$

with $x'_a = v_{xa} / v_s \approx P_{xa} / P_s$ and $y'_a = v_{ya} / v_s \approx P_{ya} / P_s$

• Suppose we want a lens system that will bring a group of such on axis particles back to the (x) axis. This could be accomplished for ALL $v_{xa}$ & $v_{ya}$ if

$$R_{12} = R_{14} = 0$$

⇒ "Point-to-Point" Focus in x  ($R_{12} = 0$)

Similarly for y:

$$R_{32} = R_{34} = 0$$

⇒ "Point-to-Point" Focus in y  ($R_{34} = 0$)

Will Return to this Later!
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont'd)

The "Standard" 6-D Phase Space Coordinates & Momenta

- Transverse coordinates are position values $x$ and $y$ perpendicular to the Reference Trajectory:
  $$[q_i] = [x_i, \ldots, y_i, \ldots, \ldots, \ldots]$$

- Transverse "momenta" are the velocity values $v_x$ and $v_y$ along $x$ and $y$, divided by the Reference Velocity $v_s$ and denoted as $x'$ and $y'$:
  $$[q_i] = [x_i, x'_i, y_i, y'_i, \ldots, \ldots]$$
  where $x' = v_x / v_s$ and $y' = v_y / v_s$

Some codes use $x' = p_x / p_s$ and $y' = p_y / p_s$, but the first-order R-matrices are the same. (Differences occur in higher order matrices or maps.)
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont'd)

The "Standard" 6-D Phase Space Coordinates & Momenta

- Longitudinal coordinate is the difference between the path length projected onto the Reference Trajectory & the Reference Path Length:
  \[ [q_i] = [x_i, x'_i, y_i, y'_i, l_i, \ldots] \]

- Longitudinal "momentum" is the momentum deviation from the Reference Momentum, divided by the Reference Momentum:
  \[ [q_i] = [x_i, x'_i, y_i, y'_i, l_i, \delta_i] \]

  where \[ \delta_i = (p_i - p_s) / p_s \]
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont'd)

- **Useful to break** \((6 \times 6)\) R-Matrix **into a set of 9 (2-by-2) submatrices:**

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{l}
\end{bmatrix} = R
\begin{bmatrix}
x_0 \\
y_0 \\
l_0
\end{bmatrix}
= \begin{bmatrix}
[R_{xx}] & [R_{xy}] & [R_{xz}] \\
[R_{yx}] & [R_{yy}] & [R_{yz}] \\
[R_{zx}] & [R_{zy}] & [R_{zz}]
\end{bmatrix}
\begin{bmatrix}
x_0 \\
y_0 \\
l_0
\end{bmatrix}
= \begin{bmatrix}
R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\
R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\
R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\
R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\
R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\
R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66}
\end{bmatrix}
\begin{bmatrix}
x_0 \\
y_0 \\
l_0
\end{bmatrix}
\]

- For many cases (drifts, quads, solenoids) only three are non-zero:

\[
\begin{align*}
R_{xx} &= \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \\
R_{yy} &= \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} \\
R_{zz} &= \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix}
\end{align*}
\]

- **This leads to a "block diagonal" R-Matrix:**

\[
R = \begin{bmatrix}
R_{xx} & 0 & 0 \\
0 & R_{yy} & 0 \\
0 & 0 & R_{zz}
\end{bmatrix}
\]

- **Bending magnets represent an exception to "block diagonal" R-Matrix**

Bends introduce dispersion: coupling between bend (x) and z,z' planes
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, ...(cont’d)

Some other matrix properties

- $R_{11}$ describes the dependence of the output $x_b$ on the input $x_a$:
  \[ R_{11} = M_x = \text{x-Magnification} \ (|R_{11}| > 1) \text{ or Demagnification} \ (|R_{11}| < 1) \]
  
  Similarly:
  \[ R_{33} = M_y = \text{y-Magnification} \ (|R_{33}| > 1) \text{ or Demagnification} \ (|R_{33}| < 1) \]

- $R_{21}$ describes the dependence of the output angle $x'_b$ on the input $x_a$:
  \[ R_{21} = \frac{-1}{f_x} \text{ where } f_x = \text{x-Focal Length} \]
  
  Similarly:
  \[ R_{43} = \frac{-1}{f_y} \text{ where } f_y = \text{y-Focal Length} \]

  $\Rightarrow R_{21} < 0$ then focusing in x direction, while $R_{21} > 0$ is defocusing in x direction
  $\Rightarrow R_{43} < 0$ then focusing in y direction, while $R_{43} > 0$ is defocusing in y direction

- For **Solenoid** or **Einzel Lens**, x and y are same ($R_{21} = R_{43}$)  $\Rightarrow$ **stigmatic lens**

- For a **Quadrupole**, x and y are not the same ($R_{21} \neq R_{43}$)  $\Rightarrow$ **astigmatic lens**
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

Not All Optics Codes Use the "Standard" Coordinates & Momenta
⇒ Comparing Results Between Different Codes Can Be Challenging!

• Example of transverse momenta differences already noted (e.g. $x' = p_x / p_s$)
• Some codes use a time-variable rather than Reference Trajectory length $s$
• Some codes use a time-of-flight variable rather than path length projection $l$
• Other differences from "Standard" $[q_i] = [x_i, x'_i, y_i, y'_i, l_i, \delta_i]$ exist
• Differences mean care must be taken in comparing maps or matrix elements
• Despite differences in $[q_i]$ definition:
  Most transverse first-order matrix (e.g. R-Matrix) elements are same
  Longitudinal first-order matrix elements are most likely different

• Physics is the same, but the differences also mean higher-order distinctions
  2nd-order (e.g. T-Matrix) in one code not exactly 2nd-order in another

⇒ PBO Lab has useful tools for matrix & map comparisons between codes
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Representations

Beam is a Collection ("Ensemble") of Particles

⇒ Can Certainly Apply Single Particle Equations of Motion to All Particles
- Some Optics Codes do this (e.g. Beamline Simulator, TURTLE)
- For Design & Other Studies Really Want:
  Computation Methods for Beam Properties
  Faster Computation than Tracking Particles

We want a method for
Describing a Beam
and procedures for simulating
the evolution of that description

⇒ Phase Space Descriptions of Beam Distributions
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- TRACE 3-D is a "1st Order" "Matrix" Code - What Does It Calculate?
  
  Does It Calculate \([q_{i\ b}] = M [q_{i\ a}] = \sum_j R_{ij} q_{ja}\)?
  
  No ⇒ TRACE 3-D Does Not Advance Individual Particles

- TRACE 3-D Advances the Beam Distribution's 1st & 2nd Moments
  - Beam Described by 1st & 2nd Moments of the Particle Distribution
  - 1st Moments of the Particle Distribution are Beam Centroids
  - 2nd Moments of the Particle Distribution are a Matrix (\(\sigma\) Matrix)

- Let a Beam Be Described by a Distribution Function \(f\):
  
  \[ f = f(x,x',y,y',z,z') \]

  with normalization:
  \[
  \int f(x,x',y,y',z,z') \, dx\, dx'\, dy\, dy'\, dz\, dz' = 1
  \]

- The Distribution Function \(f\) gives the Particle Density in Phase Space

- The Longitudinal Variables \((z,z')\) Are Understood to Mean \((l,\delta)\)
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- **First Moment for** $<x>$ **of the Distribution Function $f$:**
  \[
  <x> = \int x \, f(x,x',y,y',z,z') \, dx \, dx' \, dy \, dy' \, dz \, dz'
  \]

- **Similar Definitions for** $<x'>$, $<y>$, $<y'>$, $<l>$, $<\delta>$

- **The Beam Centroid Vector** $[q_i]_c$ **is Given by 1\textsuperscript{st} Moments:**
  \[
  [q_i]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = (<x>, <x'>, <y>, <y'>, <l>, <\delta>)
  \]

- **If the Beam Centroid Follows the Reference Trajectory Then**
  \[
  [q_i]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = 0
  \]

- **Reference Trajectory = Optical Component "Central" Axis**
  \[\Rightarrow\text{Fields are Expanded About that Central Axis}\]

- **Beam Centroid = Beam Location with Respect to that Central Axis**
  \[\Rightarrow\text{Beam 2\textsuperscript{nd} Moments Computed with Respect to Beam Centroid}\]

  \[
  \bullet\text{ Some Works Use "Centroid" & "Reference" Trajectory Interchangeably}\]

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- Second Moments Defined by Quadratic Forms of Variables:
  \[
  \langle x^2 \rangle = \int (x)^2 f(x,x',y,y',z,z') \, dx \, dx' \, dy \, dy' \, dz \, dz'
  \]
  where we assume that centroid has been removed (\( \langle x^2 \rangle \equiv \langle (x-x_c)^2 \rangle \))

- Again, Similar Definitions for \( \langle xx' \rangle, \langle xy \rangle, \langle xy' \rangle, \langle xI \rangle, \langle x\delta \rangle, \ldots \)

- Second Moments Can Be Written as a 6-by-6 Matrix, the \( \sigma \) Matrix:

\[
\sigma_{ij} = \begin{bmatrix}
\langle x^2 \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle & \langle xz \rangle & \langle xz' \rangle \\
\langle xx' \rangle & \langle x'^2 \rangle & \langle x'y \rangle & \langle x'y' \rangle & \langle x'z \rangle & \langle x'z' \rangle \\
\langle xy \rangle & \langle x'y \rangle & \langle y^2 \rangle & \langle y' \rangle & \langle y'z \rangle & \langle y'z' \rangle \\
\langle xy' \rangle & \langle x'y' \rangle & \langle y' \rangle & \langle y'^2 \rangle & \langle y'z \rangle & \langle y'z' \rangle \\
\langle xz \rangle & \langle x'z \rangle & \langle y'z \rangle & \langle z^2 \rangle & \langle z \rangle & \langle z' \rangle \\
\langle xz' \rangle & \langle x'z' \rangle & \langle y'z' \rangle & \langle z' \rangle & \langle z'^2 \rangle & \langle z' \rangle \\
\end{bmatrix}
\]

- The \( \sigma \) Matrix, aka "Beam Matrix", is Symmetric (e.g. \( \langle xx' \rangle = \langle x'x \rangle \))

- If Particle Coordinates Transform as \( [q_{ib}] = \Sigma_j R_{ij} \, q_{ja} \equiv R[q_{ia}] \)
  It Can Be Shown that the Sigma Matrix \( [\sigma_{ib}] \) Transforms as:

\[
[\sigma_{ib}] = \Sigma_k R_{ik} \Sigma_m R_{mj} \, [\sigma_{km}] \equiv R[\sigma_{ia}] R^T
\]

  where \( R^T \) is the Transpose of \( R \).
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- The $\sigma$ Matrix is Symmetric $\Rightarrow$ Use a "Reduced" Representation
- Define Correlation Parameters by:
  \[
  r_{12} \equiv r_{xx'} = \frac{\langle xx' \rangle}{\sqrt{\langle xx \rangle \langle x'x' \rangle}} = \frac{\sigma_{xx'}}{\sigma_{xx} \sigma_{x'x'}}^{1/2}
  \]
  \[
  r_{13} \equiv r_{xy} = \frac{\langle xy \rangle}{\sqrt{\langle xx \rangle \langle yy' \rangle}} = \frac{\sigma_{xy}}{\sigma_{yy} \sigma_{xy}}^{1/2}
  \]
  etc. for the complete set of $r_{ij}$ for $1 \leq i,j \leq 6$
- From $\sigma$-matrix properties: $r_{ij} \equiv r_{ji}$ and $r_{ii} \equiv 1$ all unitless
- Can Write the "Reduced" $\sigma$-Matrix as a Lower Half-Matrix:
  \[
  \sigma_{\text{reduced}} = \begin{bmatrix}
  \langle x^2 \rangle^{1/2} & \langle x'^2 \rangle^{1/2} & r_{21} \\
  \langle y^2 \rangle^{1/2} & \langle y'^2 \rangle^{1/2} & r_{31} & r_{32} \\
  \langle z^2 \rangle^{1/2} & r_{41} & r_{42} & r_{43} \\
  \langle z'^2 \rangle^{1/2} & r_{51} & r_{52} & r_{53} & r_{54} \\
  & r_{61} & r_{62} & r_{63} & r_{64} & r_{65}
  \end{bmatrix}
  \]
- The First Column Contains the RMS Beam Envelopes ("Sizes")
  - PBO Lab Beam Piece gives UNITS Immediately After 1st Column
- There Are 15 Correlation Coefficients ($r_{ij}$ with $i<j$, and $j$ running from 1 to 5)
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- Similar to the R-Matrix, the Sigma-Matrix is Often "Block Diagonal"
- Can Then Write the $\sigma$-Matrix as the Three (non-zero) Submatrices:

$$
\sigma = \begin{bmatrix}
\sigma_{xx} & 0 & 0 \\
0 & \sigma_{yy} & 0 \\
0 & 0 & \sigma_{zz}
\end{bmatrix}
$$

- Where Each Submatrix is a 2×2 Matrix:

$$
\sigma_{xx} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{bmatrix} \quad \sigma_{yy} = \begin{bmatrix}
\sigma_{33} & \sigma_{34} \\
\sigma_{43} & \sigma_{44}
\end{bmatrix} \quad \sigma_{zz} = \begin{bmatrix}
\sigma_{55} & \sigma_{56} \\
\sigma_{65} & \sigma_{66}
\end{bmatrix}
$$

- Symmetry of $\sigma$-Matrix (e.g. $\sigma_{21} = \sigma_{12}$) Means 3 Independent Parameters for Each 2×2 Matrix. So, if it Proves Useful, Can Write Each in Form such as:

$$
\sigma_{xx} = \varepsilon_x \begin{bmatrix}
\beta_x & -\alpha_x \\
-\alpha_x & \gamma_x
\end{bmatrix} \quad \text{with } \beta_x \gamma_x - \alpha_x^2 = 1
$$
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- **Motion of a particle moving under linear restoring force (harmonic oscillator)** can be described in terms of **amplitude and phase** variables by:

\[
x(s) = \left[\beta_x \epsilon_x\right]^{1/2} \cos(\psi_x(s) + \psi_x(0))
\]

where **\(\beta_x\) is constant and \(\psi_x(s)\) is linear in \(s\):**

\[
\beta_x = \frac{x(0)^2}{\epsilon_x}, \quad \psi_x(s) = k_x s \quad \text{(think of \(s\) as \(t\) or \(z\)}
\]

- **\(\beta_x\) is the amplitude and \(\psi_x(s)\) is the phase**

- **As \(s\) increases the particle traces an ellipse in Phase Space**

- **If the force "constant" \(k_x\) is not a constant, but changes with \(s\), e.g. \(k_x(s)\), the motion can still be described in terms of amplitude and phase:**

\[
x(s) = \left[\beta_x(s) \epsilon_x\right]^{1/2} \cos(\psi_x(s) + \psi_x(0))
\]

- **Now \(\beta_x(s)\) is not constant and is \(\psi_x(s)\) nonlinear:**

\[
d\beta_x(s)/ds = -2\alpha_x(s) \quad \psi_x(s) = \int [\beta_x(s)]^{-1} \, ds
\]
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- The function $\beta_x(s)$ is referred to as the amplitude function
- The function $\psi_x(s)$ is referred to as the phase advance
- As $s$ increases the particle will still remain on the ellipse, but now the ellipse will change in phase space

![Diagram of phase space with ellipses showing evolution](image)

- If you start with an collection of particles ("beam") with a set of initial amplitudes and phases inside of a given ellipse the ellipse will evolve ("down the beamline") and all particles will remain within that ellipse.

$\Rightarrow$ Representation Provides a Useful Way to Describe a Beam (at least for linear optics)
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont’d)

Relation Between $\sigma$-Matrix (Semi-Axis) Parameters and $\alpha$-$\beta$ (Twiss) Parameters

$$\beta_x \varepsilon_x \equiv \sigma_{11} = \langle x^2 \rangle = (x_{\text{max}})^2$$

$$\gamma_x \varepsilon_x \equiv \sigma_{22} = \langle x'^2 \rangle = (x'_{\text{max}})^2$$

$$\alpha_x \varepsilon_x \equiv \sigma_{12} = \langle xx' \rangle = r_{12} \left[ \sigma_{11} \sigma_{22} \right]^{1/2} \quad \text{or} \quad \alpha_x = -1 / \left[ 1 - r_{12}^2 \right]^{1/2}$$

$$r_{12} \equiv \sigma_{12}/[\sigma_{11} \sigma_{22}]^{1/2} = r_{21} \equiv \sigma_{21}/[\sigma_{11} \sigma_{22}]^{1/2}$$
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

Twiss Representation

- The semi-major axis is given by $x'_m = \left[ \gamma_x \varepsilon_x \right]^{1/2}$.
- The semi-minor axis is given by $x'_e = -\alpha_x \left[ \varepsilon_x / \beta_x \right]^{1/2}$.

**Geometric Parameterization**

- The semi-major axis is given by $x_m = \left[ \beta_x \varepsilon_x \right]^{1/2}$.
- The semi-minor axis is given by $x_j = \left[ \varepsilon_x / \gamma_x \right]^{1/2}$.
- The angle $\Theta_x$ is defined by:
  \[
  \gamma_x / \varepsilon_x = (\cos \Theta_x / A_x)^2 + (\sin \Theta_x / B_x)^2
  \]
  \[
  \beta_x / \varepsilon_x = (\sin \Theta_x / A_x)^2 + (\cos \Theta_x / B_x)^2
  \]
  \[
  \alpha_x / \varepsilon_x = \cos \Theta_x \sin \Theta_x \left[ (1 / B_x)^2 - (1 / A_x)^2 \right]
  \]
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont’d)
Note: Can Transform Ellipse into a Circle ⇒ Beam Comparisons (TRACE 3-D)

\[ \beta_x \varepsilon_x = (a_x)^2 \]

\[ \tan \chi_x = (\tan \psi_x - \alpha_x) / \beta_x \]

alternatively: \[ \cos \psi_x = \cos \chi_x / [(\cos \chi_x)^2 + (\alpha_x \cos \chi_x - \beta_x \sin \chi_x)^2]^{1/2} \]
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- The parameters $\alpha_x, \beta_x$ and $\gamma_x$ are often called "Twiss Parameters"
  ⇒ I (try to) use the term Twiss Parameters when describing beam properties

- The $\alpha_x, \beta_x$ and $\gamma_x$ are also called the "Courant-Snyder Parameters"
  ⇒ I use Courant-Snyder Parameters when describing machine properties

- For a periodic (stable) system, the functions $\alpha_x(s)$ and $\beta_x(s)$ will be periodic

Example of $\beta_x(s)$ and $\beta_y(s)$ for a storage ring

- The only beam that will survive many passes through such a system (i.e. many turns around the ring) is called the "matched beam"
  ⇒ The "matched beam" requirements are machine properties
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont’d)

- Two slits will define an "acceptance" phase space for a beam
- Slit-defined "acceptance" phase space also specifies an enclosing ellipse

Phase space accepted by two slits, each of width W, separated by L.
Phase space accepted by three slits, with focusing elements in between.

[Figures from page 101 of *The Optics of Charged Particle Beams* by David C. Carey, Harwood Academic Publishers (1987).]

- An inscribed "acceptance" ellipse can be used to describe accepted beam
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont’d)

**Emittances: RMS, Boundary, Normalized, Unnormalized, …**

⇒ The value of $\varepsilon_x$, which is a measure of the ellipse area, is the \textit{x-emittance}.

One convention defines $\varepsilon_x$ as the area of the ellipse divided by $\pi$, and written as

$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x'x \rangle^2} = 0.1 \ \pi\text{-mm-mrad} \quad (\pi \ \text{in the units})$$

- Emittance Definitions & Usage are \textbf{Not Uniform}
- Laboratory Emittance = "unnormalized" emittance (assumption so far)
- Emittance ellipse may describe all, or only a part, of the beam

![Enclosing Ellipse: "boundary" emittance](image1)

![Partially Enclosing Ellipse: "RMS" emittance, "50% contour" emittance, ...](image2)
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont’d)

**Emittances: RMS, Boundary, Normalized, Unnormalized, …**

\[ \varepsilon_x = [\langle x^2 \rangle \langle x'^2 \rangle - \langle x'x \rangle^2]^{1/2} \; \pi\text{-mm-mrad} \quad (\pi \text{ in the units}) \]

- **Laboratory emittance (i.e. "unnormalized")** not preserved with acceleration

  \[ \varepsilon_x(E_b) = \varepsilon_x(E_a) \left[ \beta_a \gamma_a \right] / \left[ \beta_b \gamma_b \right] \]

  ⇒ **These** \( \beta_a \gamma_a \) and \( \beta_b \gamma_b \) are relativistic velocity and energy parameters, **not Twiss!**

- **Normalized emittance,** \( \varepsilon_{x,n} \), **is preserved with acceleration**

  \[ \varepsilon_{x,n} = [\beta_a \gamma_a] \varepsilon_x(E_a) \]

- **Laboratory emittance scales with square root of kinetic energy** \( E_a \rightarrow E_b \)

  \[ \varepsilon_x(E_b) = \varepsilon_x(E_a) \left[ \beta_a \gamma_a \right] / \left[ \beta_b \gamma_b \right] \approx \varepsilon_x(E_a) \left[ v_a/v_b \right] \approx \varepsilon_x(E_a) \left[ E_a / E_b \right]^{1/2} \]

- **Reported emittance may also need to be scaled with mass** \( m_a \rightarrow m_b \)

  \[ \varepsilon_x(m_b) = \varepsilon_x(m_a) \left[ \beta_a \gamma_a \right] / \left[ \beta_b \gamma_b \right] \approx \varepsilon_x(m_a) \left[ E_a / E_b \right]^{1/2} \left[ m_b / m_a \right]^{1/2} \]

[Often emittances from ion source are different for different mass particles.]
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

**Emittances: RMS, Boundary, Normalized, Unnormalized, …**

- **Example:** KACST ion injector design acceptance:

- **Acceptance** $\varepsilon_{\text{acceptance}}$ of beamline for 30 kV extraction is:

  $$\varepsilon_{\text{acceptance}} = 120 \ \pi\text{-mm-mrad}$$

  [from Table 1 of "A Versatile Ion Injector at KACST," M. O. A. El Ghazaly, S. A. Behery, A. A. Almuqhim, A. I. Papish and C. P. Welsch, AIP Conference Proceedings 1370, 272-277 (2011).]

- **Normalized or Un-Normalized?** If We ("I") Assume Normalized by $\beta\gamma$:

- Extraction energy will be $E_a = 30\times q$ keV where $q$ is the charge state of the ion

- The $\beta, \gamma$ values for a (30 keV) proton ($q = 1$) are $\beta_a = 0.007997$ and $\gamma_a = 1.000032$

- The equivalent laboratory emittance to use in calculations is thus:

  $$\varepsilon_{\text{lab}}(30 \text{ keV } H^+) = \varepsilon_{\text{acceptance}} / (\beta_a \gamma_a) = 120 \ \pi\text{-mm-mrad} / 0.007997 = 15005. \ \pi\text{-mm-mrad}$$

- For 30 kV charge 2 ($q = 2$) heavy (1500 amu) ion the emittance for calculations:

  $$\varepsilon_{\text{lab}}(60 \text{ keV } 1500 \text{ amu}) = 15005 \ \pi\text{-mm-mrad} [(30.0/60.0)^{1/2}(1500/1.007276)^{1/2}]$$
  $$= 15005 \times (0.7071 \times 38.59) \ \pi\text{-mm-mrad} = 409442 \ \pi\text{-mm-mrad}$$

  [Using $\beta_b = 0.000293$ and $\gamma_b = 1.000000$ the direct calculation gives same result:

  $$\varepsilon_{\text{lab}}(60 \text{ keV } 1500 \text{ amu}) = \varepsilon_{\text{acceptance}} / (\beta_b \gamma_b) = 120/0.000293 \ \pi\text{-mm-mrad} = 409556 \ \pi\text{-mm-mrad}$$

  ⇒ These laboratory emittances seem "large" ⇒ Assumption probably wrong.
2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. ...(cont'd)

- Virtually All Optics Codes (including TRACE 3-D) Use Laboratory Emittance
- TRACE 3-D Uses an "Equivalent Uniform Beam" Model of A Beam

Enclosing Ellipse Gives Boundary "bnd" Emittance $\equiv bnd$ Emittance $= 5 \times$ RMS Emittance

Enclosing Ellipse Gives For a Bunched (3-D) Beam

For Continuous (2-D or DC) Beam

- PBO Lab Uses the TRACE 3-D Bunched Beam Convention: $\varepsilon_{bnd} = 5\varepsilon_{rms}$
- Beam "Sizes" $(5)^{1/2} \times$ RMS Sizes: $\langle x^2 \rangle = (x_{max})^2$ where $x_{max} = (2.2360...) \times x_{rms}$
3. Equations of Motion: Drifts, Quads, Bends

But how do we get from $F = ma$ to the matrix formalism?

⇒ Equations of Motion
3. Equations of Motion: Drifts, Quads, Bends

- Classical mechanics, Newton's 2nd Law:
  \[ F = \frac{dp}{dt} \quad (F, p \text{ 3-vectors}) \]
- Relativistically correct, with proper interpretation of \( F \) and \( p \) (need a 4-vector)
  - Spatial components:
    \[ F_x = \frac{dp_x}{dt} \quad \text{with} \quad p_x = \beta_x \gamma mc \]
    \[ F_y = \frac{dp_y}{dt} \quad \text{with} \quad p_y = \beta_y \gamma mc \]
    \[ F_z = \frac{dp_z}{dt} \quad \text{with} \quad p_z = \beta_z \gamma mc \]
  - 4th component (energy \( W \)):
    \[ F \cdot v = \frac{dW}{dt} \quad \text{with} \quad W^2 = p^2 c^2 + m^2 c^4 \]
- More elegant formulation uses Hamiltonian mechanics (not discussed further here)

Equations of Motion
\[ F = ma \text{ Version} \]

This section of the lecture uses:

Bold Font for 3-Vectors
Plain Font for Scalars

The \( z \) coordinate is often denoted by \( l \)
\[ l = \text{path length difference} \]
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion (con't)

- For cases where the Reference Trajectory is straight, and there is no acceleration (i.e. all magnetic elements except bends), the equations of motion in TRACE 3-D coordinates can be derived using the relation:

\[
\frac{d}{dt} = \left(\frac{ds}{dt}\right) \frac{d}{ds} \equiv c\beta_s \frac{d}{ds}, \quad \text{e.g. } v_x \equiv \frac{dx}{dt} = c\beta_s \frac{dx}{ds} \equiv c\beta_s x'
\]

- Transverse motion (x and y):

\[
F_x = \frac{dp_x}{dt} = c\beta_s \frac{d}{ds} \left(\beta_x \gamma mc\right) = c\beta_s \frac{d}{ds} (x'\beta_s \gamma mc) = c\beta_s^2 \gamma mc \frac{dx'}{ds}
\]
\[
F_y = \frac{dp_y}{dt} = c\beta_s \frac{d}{ds} \left(\beta_y \gamma mc\right) = c\beta_s \frac{d}{ds} (y'\beta_s \gamma mc) = c\beta_s^2 \gamma mc \frac{dy'}{ds}
\]

where: \( x' = \frac{dx}{ds}, \quad y' = \frac{dy}{ds} \)

- Convenient to write these in the form:

\[
\frac{dx}{ds} = x' \quad \frac{dx'}{ds} = \left[\frac{F_x}{p_s}\right] \frac{1}{(c\beta_s)} \left(\gamma_s/\gamma\right)
\]
\[
\frac{dy}{ds} = y' \quad \frac{dy'}{ds} = \left[\frac{F_y}{p_s}\right] \frac{1}{(c\beta_s)} \left(\gamma_s/\gamma\right)
\]

- Use the Lorentz force to get the forces \( F_x, F_y \) for particular fields

For a force free region (e.g. drift space) \( F_x = F_y = 0 \), hence \( \frac{dx'}{ds} = \frac{dy'}{ds} = 0 \)

So that \( \frac{dx}{ds} = x' = \text{constant}_x, \) and \( \frac{dy}{ds} = y' = \text{constant}_y \)
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion (con't)

- Longitudinal motion in TRACE 3-D coordinates ($l$ and $\delta$), when the Reference Trajectory is straight and there is no acceleration (i.e. all magnetic elements except bends), is simple but non-trivial

- For magnetic fields, where $F = q (v \times B)$, then $F \cdot v = 0$, and $dW/dt = 0$
  
  \[ dW/dt = c\beta_s \text{ and } dW/ds = c\beta_s \frac{d(\gamma mc^2)}{ds} = mc^3\beta_s \frac{d\gamma}{ds} = 0 \]

  If $d\gamma/ds = 0$, then $d\beta/ds = 0$ also, and likewise $d(\beta\gamma)/ds = 0$

- So with no acceleration, then for the Reference Trajectory variables

  \[ d(\beta, \gamma)/ds = 0 \]

  and since $\delta = \left[ \frac{\beta\gamma}{(\beta_s\gamma_s)} \right] - 1$ one then has:

  \[ d\delta/ds = 0 \quad (\Rightarrow \text{Conservation of Energy, Bends too}) \]

- More general case (e.g. with acceleration) one can show that

  \[ \frac{d\delta}{dz_s} = \frac{1}{(1+\delta)\beta_s\gamma_s} \frac{1}{c\beta_s p_s} \left[ F_z + \left( \frac{v_x}{v_z} \right) F_x + \left( \frac{v_y}{v_z} \right) F_y \right] - \frac{(1+\delta)}{\beta_s\gamma_s} \frac{d(\beta\gamma)}{dz_s} \]

- What about the longitudinal coordinate $l$?
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion (con't)

- \( l \) is the projected (on \( z \) direction) path length difference
  \[
  \frac{dl}{dt} = c\beta_s \frac{dl}{ds}, \quad \text{but} \quad \frac{dl}{dt} = v_z - v_s = c(\beta_s - \beta_z), \quad \text{hence}
  \frac{dl}{ds} = (\beta_s/\beta_z) - 1
  \]

- Longitudinal velocity \( \beta_z \) (not conserved) in terms of other variables \((x', y', \delta)\)
  \[
  \beta_z = \left\{ (\beta)^2 - (\beta_z)^2 - (\beta_s)^2 \right\}^{1/2} = \left\{ (\beta)^2 - (x'\beta_s)^2 - (y'\beta_s)^2 \right\}^{1/2}
  \]
  it can be shown that \( \beta = \beta_s (1 + \delta) / \left[ 1 + \delta(2+\delta)(\beta_s)^2 \right]^{1/2} \), hence
  \[
  \beta_z = \beta_s \left\{ (1 + \delta)^2 / \left[ 1 + \delta(2+\delta)(\beta_s)^2 \right] \right\}^{1/2} - (x')^2 - (y')^2 \right\}^{1/2}, \quad \text{so finally:}
  \frac{dl}{ds} = \left\{ ((1 + \delta)^2 / \left[ 1 + \delta(2+\delta)(\beta_s)^2 \right]) - x'^2 - y'^2 \right\}^{1/2} - 1
  \]

- Note that \( \frac{dl'}{ds} \neq 0 \), where \( l' = dl / ds \), but has an "apparent" "force" \( F_l \):
  \[
  F_l = (d/ds)\left\{ ((1 + \delta)^2 / \left[ 1 + \delta(2+\delta)(\beta_s)^2 \right]) - x'^2 - y'^2 \right\}^{1/2}
  \]

- Longitudinal coordinate \( l \) important for radiofrequency (RF) components
  
  * Not all codes use the (TRACE 3-D, TRANSPORT) longitudinal coordinate \( l \)
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion - Drift

For (non dipole) magnetic systems without acceleration, the equations of motion in TRACE 3-D variables \(x, x', y, y', l, \delta\) are:

\[
\begin{align*}
\frac{dx}{ds} &= x' \\
\frac{dx'}{ds} &= \left[\frac{F_x}{p_s}\right] \frac{1}{c\beta_s} \left\{\frac{1}{1 + \delta(2+\delta)(\beta_s)^2}\right\}^{1/2} \\
\frac{dy}{ds} &= y' \\
\frac{dy'}{ds} &= \left[\frac{F_y}{p_s}\right] \frac{1}{c\beta_s} \left\{\frac{1}{1 + \delta(2+\delta)(\beta_s)^2}\right\}^{1/2} \\
\frac{dl}{ds} &= \left\{\left[\frac{(1 + \delta)^2}{1 + \delta(2+\delta)(\beta_s)^2}\right] - x'^2 - y'^2\right\}^{1/2} - 1 \\
\frac{d\delta}{ds} &= 0
\end{align*}
\]

Return to the simplest example: Drift (field free region):

\[
\begin{align*}
\frac{dx}{ds} &= x' \\
\frac{dx'}{ds} &= 0 \\
\frac{dy}{ds} &= y' \\
\frac{dy'}{ds} &= 0 \\
\frac{dl}{ds} &= \left\{\left[\frac{(1 + \delta)^2}{1 + \delta(2+\delta)(\beta_s)^2}\right] - x'^2 - y'^2\right\}^{1/2} - 1 \\
\frac{d\delta}{ds} &= 0
\end{align*}
\]

Integrate \(ds\) from \(s_a\) to \(s_b\), with \(s_b-s_a = L\), the length of the drift, the solutions are:

\[
\begin{align*}
x_b &= x_a + x'_a L \\
y_b &= y_a + y'_a L \\
l_b &= l_a + \left\{\left[\frac{(1 + \delta_o)^2}{1 + \delta_o(2+\delta_o)(\beta_o)^2}\right] - x'^2_a - y'^2_a\right\}^{1/2} L - L \\
\delta_b &= \delta_a \equiv \delta_o
\end{align*}
\]

⇒ Drift is inherently nonlinear for the longitudinal coordinate

\[
l_b \approx l_a + \left(\delta_o/\gamma_s^2\right) L + O(\delta_o^2, x'_a^2, y'_a^2) + \text{higher order terms}
\]

[ • Although codes may not use the (TRACE 3-D, TRANSPORT) longitudinal coordinate \(l\), their equivalent longitudinal coordinates are still nonlinear]
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion - Drift (con't)

- Solution for Drift is in the form of matrix equation, taking an initial vector \([q_{ia}] = (x_o, x'_o, y_o, y'_o, l_o, \delta_o)\) to a final vector \([q_{ib}] = (x, x', y, y', l, \delta)\), where the R-Matrix equation, \(q_{ib} = [R] q_{ia}\), is obtained using only \(l = (\delta_o/\gamma^2) L + l_o\).

- Useful to break (6-by-6) R-Matrix into a set of 9 (2-by-2) submatrices:

\[
\begin{bmatrix}
  x \\
  x' \\
  y \\
  y' \\
  l \\
  \delta
\end{bmatrix}
= R
\begin{bmatrix}
  x_o \\
  x'_o \\
  y_o \\
  y'_o \\
  l_o \\
  \delta_o
\end{bmatrix}
= \begin{bmatrix}
  [R_{xx}] & [R_{xy}] & [R_{xz}] \\
  [R_{yx}] & [R_{yy}] & [R_{yz}] \\
  [R_{zx}] & [R_{zy}] & [R_{zz}]
\end{bmatrix}
\begin{bmatrix}
  x_o \\
  x'_o \\
  y_o \\
  y'_o \\
  l_o \\
  \delta_o
\end{bmatrix}
= \begin{bmatrix}
  R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\
  R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\
  R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\
  R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\
  R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\
  R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66}
\end{bmatrix}
\begin{bmatrix}
  x_o \\
  x'_o \\
  y_o \\
  y'_o \\
  l_o \\
  \delta_o
\end{bmatrix}
\]

- For a drift, most submatrices are zero, only three are non-zero:

\[
R_{xx} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \quad R_{yy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \quad R_{zz} = \begin{bmatrix} 1 & s/\gamma^2 \\ 0 & 1 \end{bmatrix}
\]

- For a drift of length \(L\), then the R-Matrix above is evaluated for \(s = L\)

- Easy to show that, for two drifts of lengths \(L_1\) and \(L_2\), the multiplication of the two R-Matrices is simply a R-Matrix for length \(L = L_1 + L_2\)
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

**R-Matrix Example - Drift**

- 250 keV protons ($\beta = 0.023080$ and $\gamma = 1.000266$)
- Drift Length of 2 Meter

The non-trivial sub matrices for the Drift Piece are determined by the value of the Effective Drift Length $L$ in the Piece Window, and the relativistic energy $\gamma$. These submatrices are:

\[
\begin{bmatrix}
R_{xx}
\end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 2.0000 \\ 0.0000 & 1.0000 \end{bmatrix}
\]

\[
\begin{bmatrix}
R_{yy}
\end{bmatrix} = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 2.0000 \\ 0.0000 & 1.0000 \end{bmatrix}
\]

\[
\begin{bmatrix}
R_{zz}
\end{bmatrix} = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} \begin{bmatrix} 1 & L\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 1.9989 \\ 0.0000 & 1.0000 \end{bmatrix}
\]
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

**Equations of Motion - Magnetic Quadrupole**

- Magnetic quadrupole is one example of magnetic cylindrical multipole field

**Magnetic Quadrupole Field:**

- Inside: \( B = \left( \frac{B_o}{a} \right) \left[ (r \sin \theta) x + (r \cos \theta) y \right] \) \((0 < z < L)\) 
- Outside: \( B = 0 \) \((z < 0 \text{ or } z > L)\)

- Lorentz Force Gives \( F_x \) and \( F_y \): \( F = (q) \left[ E + v \times B \right] = (q) v \times B \)
3. Equations of Motion: Drifts, Quads, Bends …(cont’d)

Equations of Motion - Magnetic Quadrupole (con't)

Key Parameters

For Optics:
Pole Tip Field, $B_o$
Bore Radius, $a (= r_1)$
Effective Length, $L$

For Engineering:
No. of Turns/Coil, $n$
Current in Coil, $I$
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

To Relate Engineering & Optics Parameters: Use Ampere's Law

Separate Integral into Four Parts

\[ \oint H \cdot ds = I(1) + I(2) + I(3) + I(4) \]

I(1) **Main Contribution from Bore Radial Integral**

\[ I(1) = \int_{r_1}^{r_1} H_{pt} dr = \int_{0}^{r_1} (B_o / \mu_o) (r/r_1) dr = B_o r_1/(2 \mu_o) \]

I(2)+ I(3) **Contributions of Pole and Yoke Small (\mu >>1)**

\[ I(2) = \int_{r_1}^{r_{int}} H_{pole} \cdot dr = (1/\mu_{pole}) \int_{r_1}^{r_{int}} B_{pole} \cdot dr \]

\[ I(3) = (r_{int}) \int_{0}^{\pi/4} H_{yoke} \cdot d\Theta = (1/\mu_{yoke}) (r_{int}) \int_{0}^{\pi/4} B_{yoke} \cdot d\Theta \]

I(4) **Contribution Vanishes** \( (H \perp dr) \)

\[ I(4) = \int_{r_{int}}^{0} H \cdot dr = 0 \quad \Rightarrow \quad B_O = 2\mu_o nI / r_1 \]
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

- **Magnetic Field:** \( B = (B_o/a) [(r \sin \theta) \ x + (r \cos \theta) \ y] = (B_o/a) [(y) \ x + (x) \ y] \), or
  \[
  B_x = (B_o/a) \ y = B' \ y \quad \text{and} \quad B_y = (B_o/a) \ x = B' \ x
  \]

- **Force Components:** \( F_x = - (qc) \ \beta_z B_y \) and \( F_y = (qc) \ \beta_z B_x \), or
  \[
  F_x = - (qcB') \ \beta_z x \quad \text{and} \quad F_y = (qcB') \ \beta_z y
  \]

- **Equations of Motion**

  \[
  \frac{dx}{ds} = x' \quad \frac{dx'}{ds} = - \frac{[(qc) \ \beta_z B'/p_s]}{(c\beta_s)} \frac{1}{(\gamma_s/\gamma)} \ x = - \frac{[qB'/p_s]}{(\beta_z/\beta_s)(\gamma_s/\gamma)} x
  \]

  \[
  \frac{dy}{ds} = y' \quad \frac{dy'}{ds} = + \frac{[(qc) \ \beta_z B'/p_s]}{(c\beta_s)} \frac{1}{(\gamma_s/\gamma)} \ y = + \frac{[qB'/p_s]}{(\beta_z/\beta_s)(\gamma_s/\gamma)} y
  \]

  \[
  \frac{dl}{ds} = (\beta_z/\beta_s)-1 \quad \frac{d\delta}{ds} = 0 \quad \text{(longitudinal same as a drift)}
  \]

- **Expand non-linear terms** \((\beta_z/\beta_s) \equiv 1 + (\delta_o/\gamma_s^2) + \ldots, \ (\beta_z/\beta_s)(\gamma_s/\gamma) \equiv 1 - \delta_o + \ldots\)

  \[
  \frac{dx}{ds} = x' \quad \frac{dx'}{ds} = - \frac{[qB'/p_s]}{(K_1)} x = - [K_1] x \quad \text{K}_1 \text{ is Quad Coefficient:}
  \]

  \[
  \frac{dy}{ds} = y' \quad \frac{dy'}{ds} = + \frac{[qB'/p_s]}{(K_1)} y = + [K_1] x
  \]

  \[
  \frac{dl}{ds} = \delta_o/\gamma_s^2 \quad \frac{d\delta}{ds} = 0 \quad \text{(longitudinal same as a drift)}
  \]
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole  (con't)

- **Transverse motion**

  \[
  \frac{dx}{ds} = x' \quad \frac{dx'}{ds} = \frac{dx^2}{ds^2} = -[K_1] x \quad \text{or} \quad \frac{dx^2}{ds^2} + [K_1] x = 0 \\
  \frac{dy}{ds} = y' \quad \frac{dy'}{ds} = \frac{dy^2}{ds^2} = + [K_1] x \quad \text{or} \quad \frac{dy^2}{ds^2} - [K_1] y = 0 
  \]

- **First order: simple harmonic type motion, solutions depend upon the sign of \(K_1\). Useful to define \(k = |K_1|^{1/2}\), then for \(K_1 > 0\):**

  \[
  x = x_o \cos(ks) + x'_o \sin(ks)/k \\
  y = y_o \cosh(ks) + y'_o \sinh(ks)/k \\
  x' = - k x_o \sin(ks) + x'_o \cos(ks) \\
  y' = + k y_o \sinh(ks) + y'_o \cosh(ks)
  \]

  \(K_1 > 0\) is an "x-focusing" quad, and \(x\) is typically taken to be horizontal direction

- **For \(K_1 < 0\), the trigonometric and hyperbolic functions are exchanged:**

  \[
  x = x_o \cosh(ks) + x'_o \sinh(ks)/k \\
  y = y_o \cos(ks) + y'_o \sin(ks)/k \\
  x' = + k x_o \sinh(ks) + x'_o \cosh(ks) \\
  y' = - k y_o \sin(ks) + y'_o \cos(ks)
  \]

- **One phase plane is focusing (trigonometric functions) and one phase plane is defocusing (hyperbolic functions)**

  \[\Rightarrow\]  Need at least 2 quads for focusing in both transverse directions
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

**Equations of Motion - Magnetic Quadrupole (con't)**

- **The Result is a Block Diagonal R-Matrix:**

\[
R = \begin{bmatrix}
R_{xx} & 0 & 0 \\
0 & R_{yy} & 0 \\
0 & 0 & R_{zz}
\end{bmatrix}
\]

- **For an x-focusing \((K_1 > 0)\) quad, the three non-zero submatrices are:**

\[
R_{xx} = \begin{bmatrix}
\cos(ks) & \sin(ks)/k \\
-k \sin(ks) & \cos(ks)
\end{bmatrix}
\quad R_{yy} = \begin{bmatrix}
\cosh(ks) & \sinh(ks)/k \\
k \sinh(ks) & \cosh(ks)
\end{bmatrix}
\quad R_{zz} = \begin{bmatrix}
1 & s/\gamma_s^2 \\
0 & 1
\end{bmatrix}
\]

- **For a quad of length \(L\), then the R-Matrix above is evaluated for \(s = L\)**

- **A useful case is where \(kL << 0\), but \(k^2L\) is not "small", then R-Matrix is:**

\[
R_{xx} = \begin{bmatrix}
1 & 0 \\
-k^2L & 1
\end{bmatrix}
\quad R_{yy} = \begin{bmatrix}
1 & 0 \\
+k^2L & 1
\end{bmatrix}
\quad R_{zz} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[\Rightarrow \quad \text{Thin Lens approximation with focal lengths} \quad f_x = 1/(k^2L) \quad \text{and} \quad f_y = -1/(k^2L)\]
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

R-Matrix Example - Quadrupole 1: QL-100

- 250 keV protons (β = 0.023080 and γ = 1.000266)
- Length L of 6 cm, Aperture a of 1.7 cm, Pole Tip Field \( B_o \) of 0.034120 T
  (Gradient \( B' = 20.0706 \) T/m, Quadrupole Coefficient \( K_1 = 277.78 \) m\(^{-2} \), \( k = 16.667 \) m\(^{-1} \))

\[\begin{align*}
\text{The Quadrupole is an Approximate First Order Optics Element} \\
\text{The non-trivial submatrices for the Quadrupole Piece are determined by the value of the Quadrupole Coefficient} \\
K_1 = \kappa^2, \text{ the Effective Length } L, \text{ and the relativistic energy } \gamma. \text{ The submatrices for a x-focusing quadrupole are:}
\end{align*}\]

\[\begin{bmatrix}
R_{xx} & R_{xy} \\
R_{yx} & R_{yy}
\end{bmatrix} = \begin{bmatrix}
\cos(\Delta \ell) & \sin(\Delta \ell) i \kappa \\
-\kappa \sin(\Delta \ell) & \cos(\Delta \ell)
\end{bmatrix} = \begin{bmatrix}
0.5403 & 0.0505 \\
-14.0246 & 0.5403
\end{bmatrix}
\]

\[\begin{bmatrix}
R_{xx} & R_{xy} \\
R_{yx} & R_{yy}
\end{bmatrix} = \begin{bmatrix}
\cosh(\Delta \ell) & \sinh(\Delta \ell) i \kappa \\
\kappa \sinh(\Delta \ell) & \cosh(\Delta \ell)
\end{bmatrix} = \begin{bmatrix}
1.5431 & 0.0705 \\
19.5869 & 1.5431
\end{bmatrix}
\]

\[\begin{bmatrix}
R_{xx} & R_{xy} \\
R_{yx} & R_{yy}
\end{bmatrix} = \begin{bmatrix}
1 & \frac{\Delta l \gamma^2}{2} \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
1.0000 & 0.0600 \\
0.0000 & 1.0000
\end{bmatrix}
\]

\[\Rightarrow \text{Thin Lens approximation gives focal lengths } f_x = - f_y = 0.06 \text{ m} \]

\[\Rightarrow \text{Thick Lens focal lengths } f_x = -1/R_{21} = 0.0713 \text{ m and } f_y = -1/R_{43} = -0.0511 \text{ m}\]
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

R-Matrix Example - Quadrupole 2: QL-100

- 10 keV Phosphorous (β = 0.000833 and γ = 1.000000)
- Length $L$ of 6 cm, Aperture $a$ of 1.7 cm, Pole Tip Field $B_o$ of 0.037840 T
  (Gradient $B' = 22.2588$ T/m, Quadrupole Coefficient $K_1 = 277.79$ m$^{-2}$, $k = 16.667$ m$^{-1}$)

The Quadrupole is an Approximate First Order Optics Element

The non-trivial submatrices for the Quadrupole Piece are determined by the value of the Quadrupole Coefficient $K_1 = K_2$, the Effective Length $L$, and the relativistic energy $γ$. The submatrices for a $x$-focusing quadrupole are:

\[
\begin{bmatrix}
R_{xx} \\
R_{yy} \\
R_{zz}
\end{bmatrix} =
\begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22} \\
R_{33} & R_{34}
\end{bmatrix} =
\begin{bmatrix}
\cos(κL) & \sin(κL) iκ \\
-k \sin(κL) & \cos(κL)
\end{bmatrix} =
\begin{bmatrix}
0.5403 & 0.0505 \\
-14.0251 & 0.5403
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_{yy} \\
R_{xx}
\end{bmatrix} =
\begin{bmatrix}
R_{33} & R_{34} \\
R_{43} & R_{44}
\end{bmatrix} =
\begin{bmatrix}
\cosh(κL) & \sinh(κL) iκ \\
k \sinh(κL) & \cosh(κL)
\end{bmatrix} =
\begin{bmatrix}
1.5431 & 0.0705 \\
19.5879 & 1.5431
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_{zz} \\
R_{xx}
\end{bmatrix} =
\begin{bmatrix}
R_{55} & R_{56} \\
R_{65} & R_{66}
\end{bmatrix} =
\begin{bmatrix}
1 & \frac{L k γ^2}{2} \\
0 & 1
\end{bmatrix} =
\begin{bmatrix}
1.0000 & 0.0000 \\
0.0000 & 1.0000
\end{bmatrix}
\]

⇒ Thin Lens approximation gives focal lengths $f_x = -f_y = 0.06$ m
⇒ Thick Lens focal lengths $f_x = -1/R_{21} = 0.0713$ m and $f_y = -1/R_{43} = -0.0511$ m
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Some ElectroMagnetic Quadrupole (EMQ) Formulas

- **Pole Tip Magnetic Field**
  \[ B_o = 2\mu_o \frac{nI}{r_1} \]
  \[ B_o(T) = 8\pi \times 10^{-7} \frac{nI(\text{Amps})}{r_1(\text{m})} \]

- **Quadrupole Gradient**
  \[ B' = B_o / r_1 = 2\mu_o \frac{nI}{(r_1)^2} \]
  \[ B'(T/m) = 8\pi \times 10^{-7} \frac{nI(\text{Amps})}{(r_1)^2} \]

- **Quadrupole Strength**
  \[ \kappa \equiv K_1 = B' / [B\rho] \]
  \[ \kappa(\text{m}^{-2}) = 0.299792 \frac{B'(T/m)}{[p(\text{GeV})]} \]

- **Effective Length**
  \[ L = [B'(0)]^{-1} \int B'(z) \, dz \sim (0.75 - 0.97) \times (\text{physical length w/coils}) \]

- **Equivalent Thin Lens Focal Length**
  \[ f = 1 / [\kappa L ] = [B\rho]/[B'L] \]
  \[ f(\text{m}) = [p(\text{GeV})]/[0.299792 \, L(\text{m}) \, B'(T/m)] \]
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion - ElectroStatic Quadrupole

Key Parameters

For Optics:
Pole Voltage, $V_o$
Bore Radius, $a$
Effective Length, $L$

For Engineering:
Electrode Radius, $R$
Power, Vacuum Considerations

- Lorentz Force Gives $F_x$ and $F_y$: $F = (q) \left[ E + v \times B \right] = (q) E$
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Equations of Motion - ElectroStatic Quadrupole (con't)

- Electric Field: \( E = -(2V_o/a^2) [(r \cos \theta) x - (r \sin \theta) y] = (-2V_o/a^2) [(x) x - (y) y], \) or
  \[
  E_x = (-2V_o/a^2) x = - G x \quad E_y = (-2V_o/a^2) y = + G y
  \]

- Force Components: \( F_x = q(e) E_x \) and \( F_y = q(e) E_y, \) or
  \[
  F_x = -(2qV_o/a^2) x \quad F_y = (2qV_o/a^2) y
  \]

- Equations of Motion
  \[
  \frac{dx}{ds} = x' \quad \frac{dx'}{ds} = - \frac{qG}{p_s c \beta_s} x = - [K_1] x \]
  \[
  \frac{dy}{ds} = y' \quad \frac{dy'}{ds} = + \frac{qG}{p_s} y = + [K_1] x \]
  \[
  \frac{dl}{ds} = (\beta_z/\beta_s) - 1 \quad d\delta / ds = 0 \quad \text{(longitudinal same as a drift)}
  \]

- Expand non-linear terms \( (\gamma_s/\gamma) \equiv 1 - \beta_s^2 \delta_o + ... \) (here is a difference from magnetic)
  \[
  \frac{dx}{ds} = x' \quad \frac{dx'}{ds} = - \frac{qG/(p_sc\beta_s)}{x} = - [K_1] x \]
  \[
  \frac{dy}{ds} = y' \quad \frac{dy'}{ds} = + \frac{qG}{p_s} y = + [K_1] x \]
  \[
  \frac{dl}{ds} = \delta_o/\gamma_s^2 \quad d\delta / ds = 0 \quad \text{(longitudinal same as a drift)}
  \]

\( \Rightarrow \) 1st Order Equations of Motion Same as Magnetic Quad with \( K_1 = (q/p_s) [2V_o/(a^2c\beta_s)] \)

\( \Rightarrow \) Differences Between Electrostatic & Magnetic Quads Occur in Higher Orders
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

Some ElectroStatic Quadrupole (ESQ) Formulas

- **ElectroStatic (ES) Quadrupole Strength**
  \[ K_1(p) = 2qV_0/(a^2\beta p) \quad & \quad \beta = \beta(p_s, \delta) \]

- **ES Quadrupole Gradient**
  \[ G = E_0/a = 2qV_0/a^2 \]
  \[ G \text{ (Volts/m}^2\text{)} = 2 [q(e)] [V_0(\text{Volts})]\!/ [a(\text{m})]^2 \]

- **ES Quadrupole Strength**
  \[ \kappa = K_1 = (q/p_s) [2V_0/(a^2c\beta_s)] \]

- **Effective Length**
  \[ L = [G(0)]^{-1} \int G(z) \, dz \sim \text{electrode physical length} \]

- **Equivalent Thin Lens Focal Length**
  \[ f = 1 /\left[ \kappa L \right] = [p_s a^2 c \beta_s] / [2qV_0L] \]

- **Equivalent Magnetic Quadrupole Gradient**
  \[ B' = 2qV_0/(a^2c\beta_s) \]
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

What about higher order optics?

- Second-order (chromatic aberrations) are (relatively) straightforward

- Replace the Reference Momentum with the Actual Momentum (e.g. in the Quadrupole Coefficient $K_1$) and expand in $\delta$

  \[ K_1(p) = \frac{qB'}{p} = \left( \frac{p_s}{p} \right) [qB'/p_s] \equiv \left( \frac{p_s}{p} \right) K_1(p_s) \]

  \[ p \equiv (1+\delta)p_s \]

  so: \( \left( \frac{p_s}{p} \right) \approx 1 - \delta + O(\delta^2) \)

- Solution to second-order equations of motion found using the Green's function approach for solving differential equations (PBO Lab).

- Third-order requires considerably more work
  - Intrinsic third-order is independent of fringe-fields
  - Fringe-field third-order require at least four integrals (Matsuda & Wollnik)

- TRACE 3-D incorporates fringe-fields by stepping through (integrating)
  - Added fringe-field third-order terms to TRANSPORT, TURTLE
  - Added Electrostatic Quadrupole to TRANSPORT, TURTLE:
    First-, Second- and Third-order: \( K_1(p) = 2qV_0/\left(a^2\beta p\right) \) & \( \beta = \beta(p_s, \delta) \)
3. Equations of Motion: Drifts, Quads, Bends …(cont’d)

Equations of Motion - Magnetic Bend

- Reference Trajectory follows an arc → curvilinear coordinates used
- For idealized Sector Dipole, 5 of the 9 submatrices are non-zero:

\[
R_{xx} = \begin{bmatrix}
\cos(hs) & \sin(hs)/h \\
-h\sin(hs) & \cos(hs)
\end{bmatrix}, \quad R_{yy} = \begin{bmatrix}
1 & s \\
0 & 1
\end{bmatrix}, \quad R_{zz} = \begin{bmatrix}
1 & s/γ^2_s \\
0 & 1
\end{bmatrix}
\]

\[
R_{xz} = \begin{bmatrix}
0 & [1-\cos(hs)]/h \\
0 & \sin(hs)
\end{bmatrix}, \quad R_{zx} = \begin{bmatrix}
-\sin(hs) & -[1-\cos(hs)]/h \\
0 & 0
\end{bmatrix}
\]

where \( h = \frac{1}{\rho} \) and \( \rho \) is the bend radius
- Dispersion & compaction are introduced by the \( R_{xz} \) and \( R_{zx} \) submatrices
- Non-Idealized Sector Dipoles are More Common
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

**Equations of Motion - Magnetic Bend (con't)**

- Non-Idealized Sector Dipole Effects that Impact First-Order Optics:
  - Fringe Fields
  - Pole Face Rotations
  - Pole Shoe Rotations

- First Order Fringe Field Effects and Pole Face Rotations are Often Referred to a "Edge Focusing"
  - Can be Modeled with Thin Lens at Entrance/Exit

- Pole Shoe Rotations Result in Non-Uniform B Field
  - First Order Effect is Radial Derivative of B Field
  - Often Referred to as a Gradient Bend
  - Can be Modeled with a Quadrupole Field added to Dipole

- Other Deviations Can also Useful
  - Curved Pole Faces, Higher-Order Combined Function Bends, ...
3. Equations of Motion: Drifts, Quads, Bends ...(cont'd)

**R-Matrix Example - Mass Separator Bend**

- 10 keV Phosphorous ( $\beta = 0.000833$ and $\gamma = 1.000000$ )
- 90° Gradient Bend $L$ of 0.785398 m, Field Index $n$ of 1/2, Pole Tip Field $B_o$ of 0.160 T, bend radius $\rho$ of 50 cm

The Bend is an Approximate First Order Optics Element

Gradient bend properties are determined by the Field Gradient Index $n$, Central Trajectory Radius $p(=1/\lambda)$ and Central Trajectory Length $L$, with $\lambda^2=(1-n)h^2p^2/n^2$:

$$
\begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix}
= \begin{bmatrix}
\cos(hL) & \sin(hL)/h_x \\
-h_x \sin(hL) & \cos(hL)
\end{bmatrix} = \begin{bmatrix}
0.4440 & 0.6336 \\
-1.2672 & 0.4440
\end{bmatrix}
$$

$$
\begin{bmatrix}
R_{15} & R_{16} \\
R_{25} & R_{26}
\end{bmatrix}
= \begin{bmatrix}
0 & h \left[1 - \cos(hL) \right]/h^2_x \\
0 & h \sin(hL)/h_x
\end{bmatrix} = \begin{bmatrix}
0.0000 & 0.5560 \\
0.0000 & 1.2672
\end{bmatrix}
$$

$$
\begin{bmatrix}
R_{33} & R_{34} \\
R_{43} & R_{44}
\end{bmatrix}
= \begin{bmatrix}
\cos(hL) & \sin(hL)/h_y \\
-h_y \sin(hL) & \cos(hL)
\end{bmatrix} = \begin{bmatrix}
0.4440 & 0.6336 \\
-1.2672 & 0.4440
\end{bmatrix}
$$

$$
\begin{bmatrix}
R_{61} & R_{62} \\
R_{61} & R_{62}
\end{bmatrix}
= \begin{bmatrix}
-h \sin(hL) \left[1 - \cos(hL) \right]/h_x & h \left[1 - \cos(hL) \right]/h^2_x \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
-1.2672 & -0.5560 \\
0.0000 & 0.0000
\end{bmatrix}
$$

$$
\begin{bmatrix}
R_{85} & R_{86} \\
R_{85} & R_{86}
\end{bmatrix}
= \begin{bmatrix}
1 - h^2 \left[1 - \cos(hL) \right]/h_x + L^2/\gamma^2 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
1.0000 & 0.4818 \\
0.0000 & 1.0000
\end{bmatrix}
$$