Overview of Particle Beam Optics Utilized in Matrix, Envelope, and Tracking Codes: TRACE 3-D, Beamline Simulator (TRANSPORT & TURTLE)

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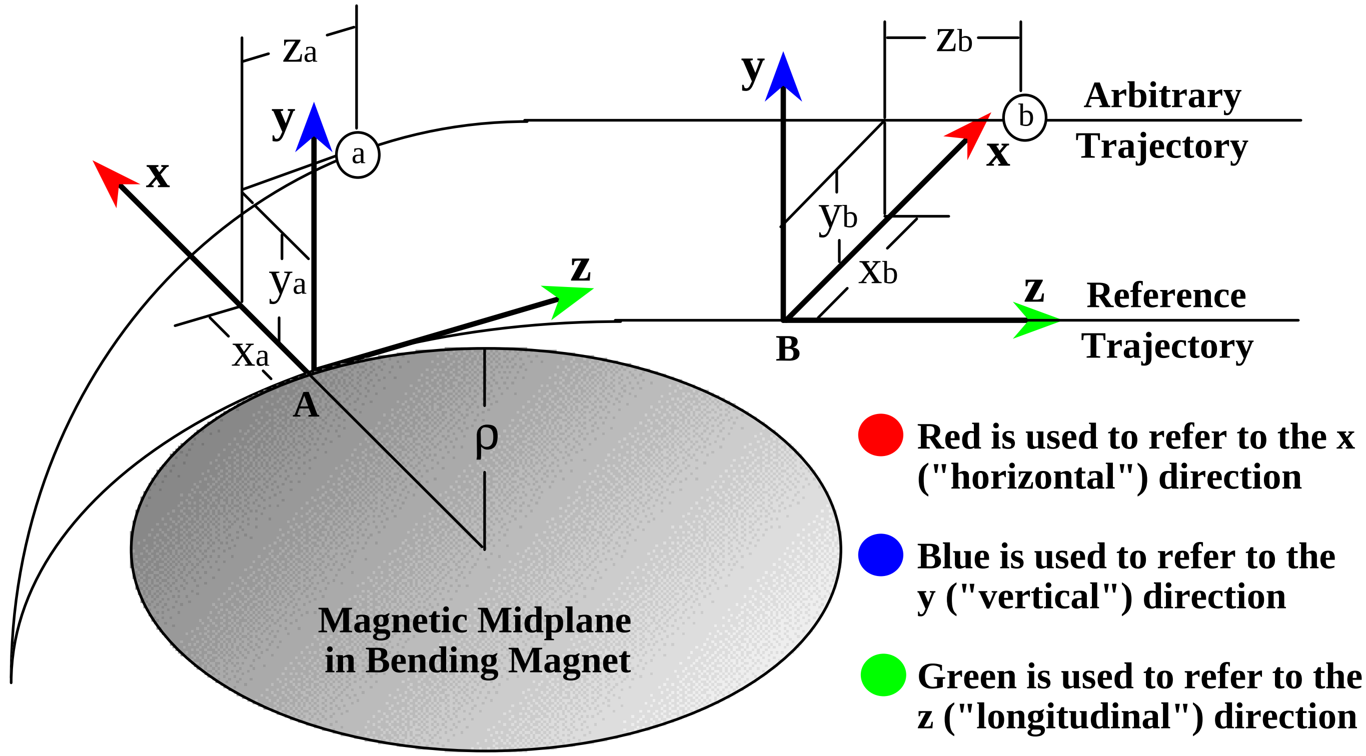
Presentation Outline - Part I

Overview of Particle Beam Optics Utilized in the Matrix, Envelope, and Tracking Codes: TRACE 3-D, Beamline Simulator (TRANSPORT & TURTLE)

Part II ⇒ **Use the PBO Lab TRACE 3-D Module to work some examples**

1. Basic Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …

- **Particle optics utilizes a perturbation approach to beam dynamics**
- **Motion measured with respect to Reference (or Synchronous) Trajectory**
- **Origin of the coordinate system moves along the Reference Trajectory**

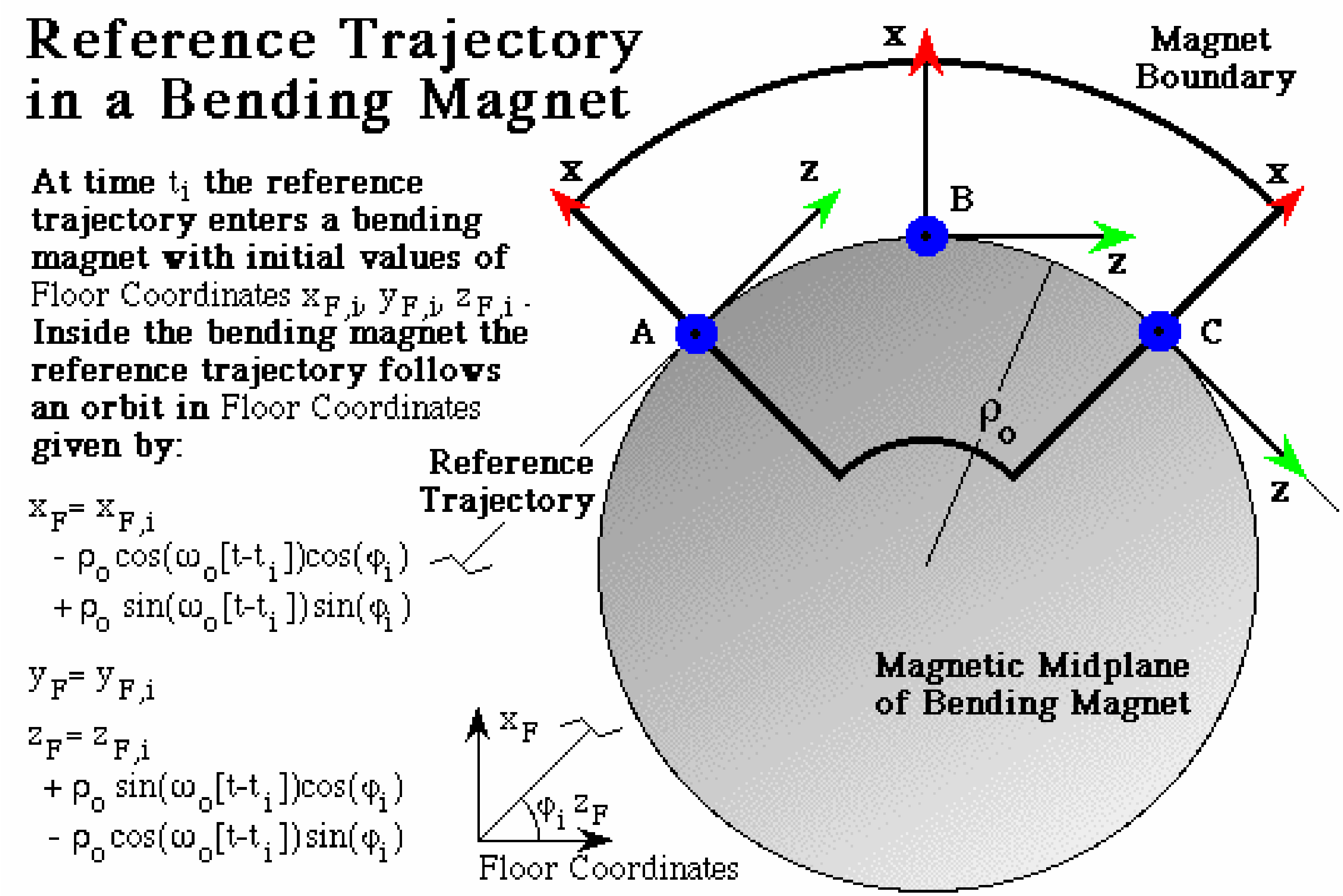


Describing Trajectories and Coordinate Systems

- **Distance along the Reference Trajectory is denoted here by** s

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

Reference Trajectory can be thought of as a machine property, often specified in terms of "floor coordinates" (denoted below by subscripts F)



1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

- **In addition to the "floor coordinates" the Reference Trajectory also has a "Reference Velocity" v_s associated with it.**
- **A "Reference Particle" (not necessarily any actual particle) moves along the Reference Trajectory at the Reference Velocity.**
- **The magnitude of the Reference (or Synchronous) Velocity is often denoted $v_s = c\beta_s$, where c is the speed of light and β_s is the relativistic speed.**
- **Similarly one can define a Reference Kinetic Energy, Reference Total Energy, Reference $\gamma_s = (1-\beta_s^2)^{-1/2}$, etc.**
- **PBO Lab Global Parameters Set Several Initial Reference Trajectory Values**
- **Some beam optics codes compute the floor coordinates of the Reference Trajectory but many do not.**

  TRACE 3-D (and TRANSPORT) compute Reference Kinetic Energy (and/or related parameters) as well as Reference Trajectory length

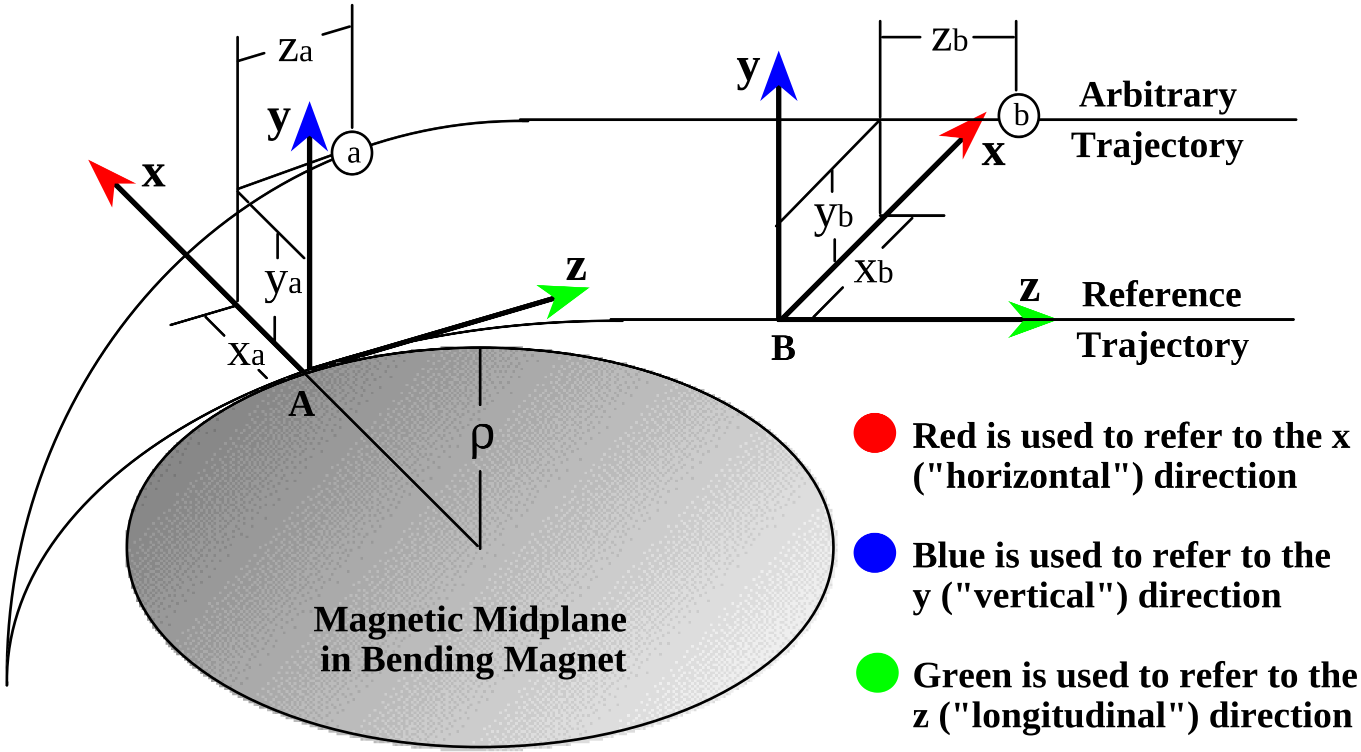
  TRACE 3-D does not compute floor coordinates

  TRANSPORT does compute floor coordinates

- **TRACE 3-D also utilizes a Reference, or Synchronous, Phase ϕ_s.**

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

So, motion measured with respect to Reference (or Synchronous) Trajectory



Describing Trajectories and Coordinate Systems

⇒ **Particle Optics:** describe $[x_b, y_b, z_b]$ in terms of $[x_a, y_a, z_a]$ as function of s

⇒ **Envelope Optics:** describe moments of a distribution $f_b(x_b, y_b, z_b)$ in terms of moments of the distribution $f_a(x_a, y_a, z_a)$ as function of s

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

- **A map, M, can be used to compute $[x_b, y_b, z_b]$ from $[x_a, y_a, z_a]$. The momentum associated with each will coordinate also be needed, e.g. $[Px_a, Py_a, Pz_a]$.**
- **If we denote the 6-vector $[x_b, Px_a, y_b, Py_a, z_b, Pz_a]$ by $[q_i]$ with i = 1,…,6 then M maps $[q_{i\,a}]$ into $[q_{i\,b}]$:**

$$[q_{i\,b}] = M\,[q_{i\,a}]$$

- **Since all elements of the 6-vectors $[q_{i\,a}]$ and $[q_{i\,b}]$ are presumed "small" we should be able to represent the map M by a Taylor series expansion:**

$$M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja} + \Sigma_{j\le k}\, \Sigma_k\, T_{ijk}\, q_{ja} q_{ka} + \Sigma_{j\le k\le l}\, \Sigma_{k\le l}\, \Sigma_l\, U_{ijkl}\, q_{ja} q_{ka} q_{la} + \ldots$$

- **In first-order optics, only the first term is used:**

$$[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja}$$

First-order optics ⇒ linear optics, described by R-matrix

- **In second-order optics, the first 2 terms are used:**

$$[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja} + \Sigma_{j\le k}\, \Sigma_k\, T_{ijk}\, q_{ja} q_{ka}$$

Second- (and higher-)order ⇒ nonlinear optics

- **Less than or equal sums (e.g. $j \le k$) avoid double counting (e.g. $T_{ijk} = T_{ikj}$).**

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

Why use this matrix formalism?

- **Consider particle starting on-axis: $x_a = 0$ and $y_a = 0$ at Reference Velocity $[v_s]$**
- **The change in the x-coordinate at the end of a system $[x_b]$ due small initial velocities $[v_{xa}, v_{ya}]$ away from the axis can be written as:**

$$x_b = R_{12}\,[x'_a] + R_{14}\,[y'_a]$$

$$\text{with}\quad x'_a = v_{xa} / v_s \approx Px_a / P_s \quad\text{and}\quad y'_a = v_{ya} / v_s \approx Py_a / P_s$$

- **Suppose we want a lens system that will bring a group of such on axis particles back to the (x) axis. This could be accomplished for ALL v_{xa} & v_{ya} if**

$$R_{12} = R_{14} = 0$$

$\Rightarrow$ **"Point-to-Point" Focus in x** $(R_{12} = 0)$

Similarly for y:

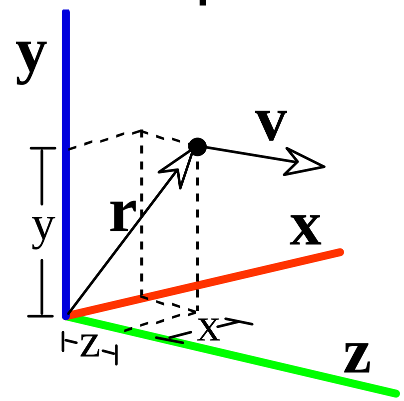
$$R_{32} = R_{34} = 0$$

$\Rightarrow$ **"Point-to-Point" Focus in y** $(R_{34} = 0)$

Will Return to this Later!

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

The "Standard" 6-D Phase Space Coordinates & Momenta



- **Transverse coordinates are position values** x **and** y **perpendicular to the Reference Trajectory:**

$$[q_i] = [x_i, \ldots, y_i, \ldots, \ldots, \ldots]$$

- **Transverse "momenta" are the velocity values** v_x **and** v_y **along x and y, divided by the Reference Velocity** v_s **and denoted as** x' **and** y'**:**

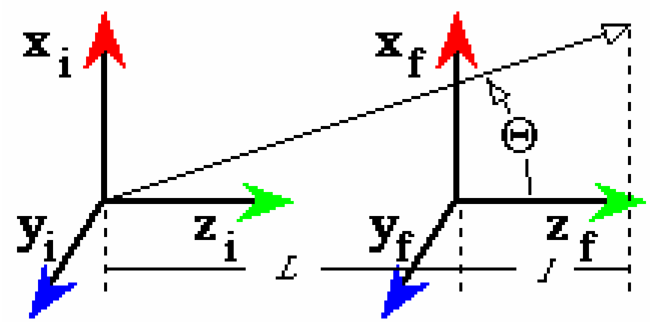
$$[q_i] = [x_i, x'_i, y_i, y'_i, \ldots, \ldots]$$

$$\text{where} \quad x' = v_x / v_s \quad \text{and} \quad y' = v_y / v_s$$

Some codes use $x' = p_x / p_s$ **and** $y' = p_y / p_s$**, but the first-order R-matrices are the same. (Differences occur in higher order matrices or maps.)**

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

The "Standard" 6-D Phase Space Coordinates & Momenta



- **Longitudinal coordinate is the difference between the path length projected onto the Reference Trajectory & the Reference Path Length:**

$$[\mathbf{q_i}] = [x_i, x'_i, y_i, y'_i, l_i, \ldots]$$

- **Longitudinal "momentum" is the momentum deviation from the Reference Momentum, divided by the Reference Momentum:**

$$[\mathbf{q_i}] = [x_i, x'_i, y_i, y'_i, l_i, \delta_i]$$

where $\delta_i = (p_i - p_s) / p_s$

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

- **Useful to break (6×6)** R-Matrix **into a set of 9 (2-by-2) submatrices:**

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{bmatrix} = R \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} [R_{xx}] & [R_{xy}] & [R_{xz}] \\ [R_{yx}] & [R_{yy}] & [R_{yz}] \\ [R_{zx}] & [R_{zy}] & [R_{zz}] \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix}$$

- **For a many cases (drifts, quads, solenoids) only three are non-zero:**

$$R_{xx} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix}$$

- **This leads to a "block diagonal"** R-Matrix**:**

$$R = \begin{bmatrix} R_{xx} & 0 & 0 \\ 0 & R_{yy} & 0 \\ 0 & 0 & R_{zz} \end{bmatrix}$$

- **Bending magnets represent an exception to "block diagonal"** R-Matrix
  Bends introduce dispersion: coupling between bend (x) and z,z' planes

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

Some other matrix properties

- **R_{11} describes the dependence of the output x_b on the input x_a :**

$R_{11} = M_x$ = x-Magnification ($|R_{11}|>1$) or Demagnification ($|R_{11}|<1$)

Similarly:

$R_{33} = M_y$ = y-Magnification ($|R_{33}|>1$) or Demagnification ($|R_{33}|<1$)

- **R_{21} describes the dependence of the output angle x'_b on the input x_a :**

$R_{21} = -1 / f_x$ where f_x = x-Focal Length

Similarly:

$R_{43} = -1 / f_y$ where f_y = y-Focal Length

$\Rightarrow$ $R_{21} < 0$ then focusing in x direction, while $R_{21} > 0$ is defocusing in x direction
$\Rightarrow$ $R_{43} < 0$ then focusing in y direction, while $R_{43} > 0$ is defocusing in y direction

- **For Solenoid or Einzel Lens, x and y are same ($R_{21} = R_{43}$) $\Rightarrow$ stigmatic lens**
- **For a Quadrupole, x and y are not the same ($R_{21} \neq R_{43}$) $\Rightarrow$ astigmatic lens**

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

Not All Optics Codes Use the "Standard" Coordinates & Momenta

⇒ Comparing Results Between Different Codes Can Be Challenging!

- **Example of transverse momenta differences already noted (e.g.** $x' = p_x / p_s$**)**
- **Some codes use a time-variable rather than Reference Trajectory length s**
- **Some codes use a time-of-flight variable rather than path length projection** l
- **Other differences from "Standard" [q_i] = [**x_i, x'_i, y_i, y'_i, l_i, δ_i**] exist**
- **Differences mean care must be taken in comparing maps or matrix elements**
- **Despite differences in [q_i] definition:**
  Most transverse first-order matrix (e.g. R-Matrix**) elements are same**
  Longitudinal first-order matrix elements are most likely different
- **Physics is the same, but the differences also mean higher-order distinctions**
  2nd-order (e.g. T-Matrix**) in one code not exactly 2nd-order in another**

⇒ PBO Lab has useful tools for matrix & map comparisons between codes

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Representations

Beam is a Collection ("Ensemble") of Particles

⇒ **Can Certainly Apply Single Particle Equations of Motion to All Particles**
- **Some Optics Codes do this (e.g. Beamline Simulator, TURTLE)**
- **For Design & Other Studies Really Want:**

Computation Methods for Beam Properties
Faster Computation than Tracking Particles

We want a method for Describing a Beam and procedures for simulating the evolution of that description

⇒ **Phase Space Descriptions of Beam Distributions**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **TRACE 3-D is a "1st Order" "Matrix" Code - What Does It Calculate?**

Does It Calculate $[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j R_{ij}\, q_{ja}$?

No ⇒ TRACE 3-D Does Not Advance Individual Particles

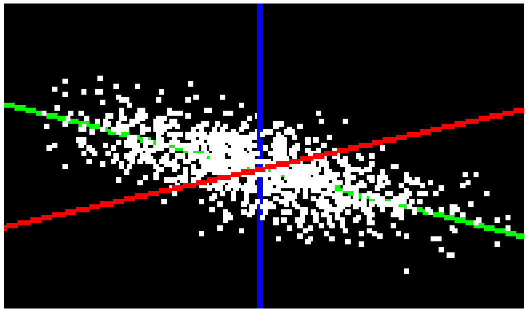
- **TRACE 3-D Advances the Beam Distribution's 1st & 2nd Moments**
  - **Beam Described by 1st & 2nd Moments of the Particle Distribution**
  - **1st Moments of the Particle Distribution are Beam Centroids**
  - **2nd Moments of the Particle Distribution are a Matrix (σ Matrix)**

- **Let a Beam Be Described by a Distribution Function f:**

$$f = f(x,x',y,y',z,z')$$

with normalization:

$$\int f(x,x',y,y',z,z')\, dxdx'dydy'dzdz' = 1$$



- **The Distribution Function f gives the Particle Density in Phase Space**
- **The Longitudinal Variables (z,z') Are Understood to Mean (l,δ)**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **First Moment for** $<x>$ **of the Distribution Function** f**:**

$$<x> = \int x\, f(x,x',y,y',z,z')\, dxdx'dydy'dzdz'$$

- **Similar Definitions for** $<x'>$, $<y>$, $<y'>$, $< l >$, $< \delta >$
- **The Beam** __**Centroid**__ __**Vector**__ $[\mathbf{q_i}]_c$ **is Given by 1st Moments:**

$$[\mathbf{q_i}]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = (<x>, <x'>, <y>, <y'>, < l >, < \delta >)$$

- **If the Beam Centroid Follows the Reference Trajectory Then**

$$[\mathbf{q_i}]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = 0$$

- **Reference Trajectory = Optical Component "Central" Axis**
  ⇒ **Fields are Expanded About that Central Axis**
- **Beam Centroid = Beam Location with Respect to that Central Axis**
  ⇒ **Beam 2nd Moments Computed with Respect to Beam Centroid**

[• **Some Works Use "Centroid" & "Reference" Trajectory Interchangeably**]

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Second Moments Defined by Quadratic Forms of Variables:**

$$\langle x^2 \rangle = \int (x)^2 \, f(x,x',y,y',z,z') \, dxdx'dydy'dzdz'$$

where we assume that centroid has been removed ($\langle x^2 \rangle \equiv \langle (x - x_c)^2 \rangle$**)**

- **Again, Similar Definitions for** $\langle xx' \rangle$, $\langle xy \rangle$, $\langle xy' \rangle$, $\langle xl \rangle$, $\langle x\delta \rangle$, **…**
- **Second Moments Can Be Written as a 6-by-6 Matrix, the σ Matrix:**

$$\sigma_{ij} = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle & \langle xz \rangle & \langle xz' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle & \langle x'y \rangle & \langle x'y' \rangle & \langle x'z \rangle & \langle x'z' \rangle \\ \langle yx \rangle & \langle yx' \rangle & \langle y^2 \rangle & \langle yy' \rangle & \langle yz \rangle & \langle yz' \rangle \\ \langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'^2 \rangle & \langle y'z \rangle & \langle y'z' \rangle \\ \langle zx \rangle & \langle zx' \rangle & \langle zy \rangle & \langle zy' \rangle & \langle z^2 \rangle & \langle zz' \rangle \\ \langle z'x \rangle & \langle z'x' \rangle & \langle z'y \rangle & \langle z'y' \rangle & \langle z'z \rangle & \langle z'^2 \rangle \end{bmatrix}$$

- **The σ Matrix, aka "Beam Matrix", is Symmetric (e.g.** $\langle xx' \rangle = \langle x'x \rangle$**)**
- **If Particle Coordinates Transform as** $[q_{i\,b}] = \Sigma_j R_{ij} q_{j\,a} \equiv R[q_{i\,a}]$
  It Can Be Shown that the Sigma Matrix $[\sigma_{ij\,b}]$ **Transforms as:**

$$[\sigma_{ij\,b}] = \Sigma_k R_{ik} \Sigma_m R_{mj} [\sigma_{km\,a}] \equiv R[\sigma_{ij\,a}]R^T$$

where R^T **is the Transpose of R.**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **The σ Matrix is Symmetric $\Rightarrow$ Use a "Reduced" Representation**
- **Define Correlation Parameters by:**

$$r_{12} \equiv r_{xx'} = \langle xx'\rangle / (\langle xx\rangle\langle x'x'\rangle)^{1/2} = \sigma_{xx'} / (\sigma_{xx}\, \sigma_{x'x'})^{1/2}$$

$$r_{13} \equiv r_{xy} = \langle xy\rangle / (\langle xx\rangle\, \langle yy'\rangle)^{1/2} = \sigma_{xy} / (\sigma_{yy}\, \sigma_{xy})^{1/2}$$

etc. for the complete set of r_{ij} for $1 \leq i,j \leq 6$

- **From σ-matrix properties:** $r_{ij} \equiv r_{ji}$ **and** $r_{ii} \equiv 1$ **all <u>unitless</u>**
- **Can Write the "Reduced" σ-Matrix as a Lower Half-Matrix:**

$$\sigma_{reduced} = \begin{bmatrix} \langle x^2\rangle^{1/2} & & & & & \\ \langle x'^2\rangle^{1/2} & r_{21} & & & & \\ \langle y^2\rangle^{1/2} & r_{31} & r_{32} & & & \\ \langle y'^2\rangle^{1/2} & r_{41} & r_{42} & r_{43} & & \\ \langle z^2\rangle^{1/2} & r_{51} & r_{52} & r_{53} & r_{54} & \\ \langle z'^2\rangle^{1/2} & r_{61} & r_{62} & r_{63} & r_{64} & r_{65} \end{bmatrix}$$

- **The First Column Contains the RMS Beam <u>Envelopes</u> ("Sizes")**
  - PBO Lab Beam Piece gives UNITS Immediately After 1st Column
- **There Are 15 Correlation Coefficients (**r_{ij} with i<j, and j running from 1 to 5**)**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Similar to the R-Matrix, the Sigma-Matrix is Often "Block Diagonal"**

- **Can Then Write the σ-Matrix as the Three (non-zero) Submatrices:**

$$\sigma = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

- **Where Each Submatrix is a 2$\times$2 Matrix:**

$$\sigma_{xx} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \qquad \sigma_{yy} = \begin{bmatrix} \sigma_{33} & \sigma_{34} \\ \sigma_{43} & \sigma_{44} \end{bmatrix} \qquad \sigma_{zz} = \begin{bmatrix} \sigma_{55} & \sigma_{56} \\ \sigma_{65} & \sigma_{66} \end{bmatrix}$$

- **Symmetry of σ-Matrix (e.g. $\sigma_{21} = \sigma_{12}$) Means 3 Independent Parameters for Each 2$\times$2 Matrix. So, if it Proves Useful, Can Write Each in Form such as:**

$$\sigma_{xx} = \varepsilon_x \begin{bmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{bmatrix}$$

with $\beta_x \gamma_x - \alpha_x^2 = 1$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

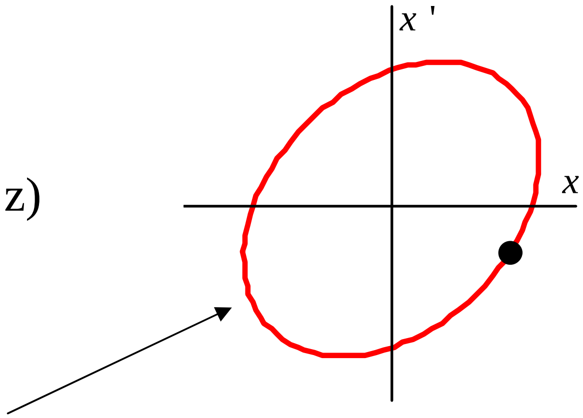
- **Motion of a particle moving under linear restoring force (harmonic oscillator) can be described in terms of amplitude and phase variables by:**

$$x(s) = \left[\beta_x \, \varepsilon_x\right]^{1/2} \cos\left(\psi_x(s) + \psi_x(0)\right)$$

where β_x is constant and $\psi_x(s)$ is linear in s:

$\beta_x = x(0)^2 / \varepsilon_x$ $\qquad \psi_x(s) = k_x s$ $\qquad$ (think of s as t or z)

- **β_x is the amplitude and $\psi_x(s)$ is the phase**
- **As s increases the particle traces an ellipse in Phase Space**



Phase Space

- **If the force "constant" k_x is not a constant, but changes with s, e.g. $k_x(s)$, the motion can still be described in terms of amplitude and phase:**

$$x(s) = \left[\beta_x(s) \, \varepsilon_x\right]^{1/2} \cos\left(\psi_x(s) + \psi_x(0)\right)$$

- **Now $\beta_x(s)$ is not constant and is $\psi_x(s)$ nonlinear:**

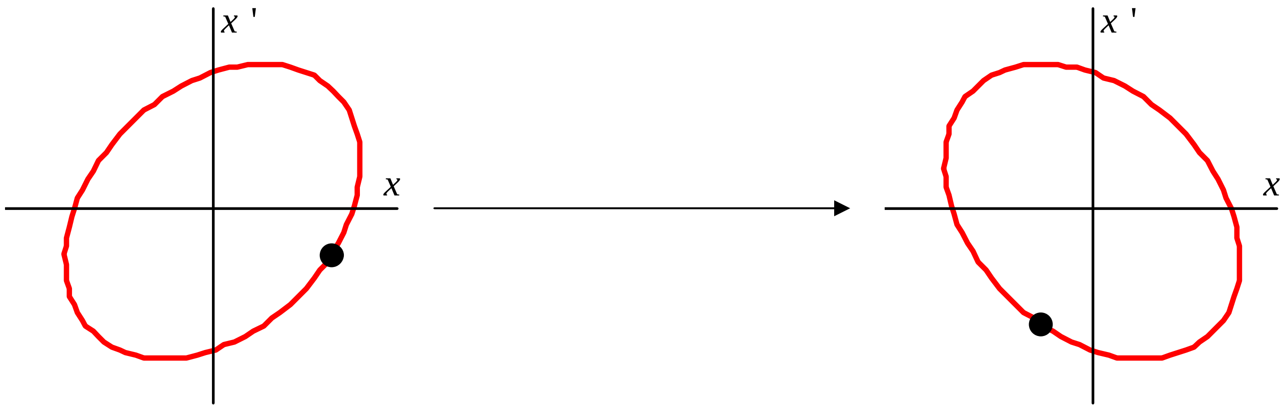
$$d\beta_x(s)/ds = -2\alpha_x(s) \qquad \psi_x(s) = \int \left[\beta_x(s)\right]^{-1} ds$$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **The function $\beta_x(s)$ is referred to as the amplitude function**

- **The function $\psi_x(s)$ is referred to as the phase advance**

- **As s increases the particle will still remain on the ellipse, but now the ellipse will change in phase space**

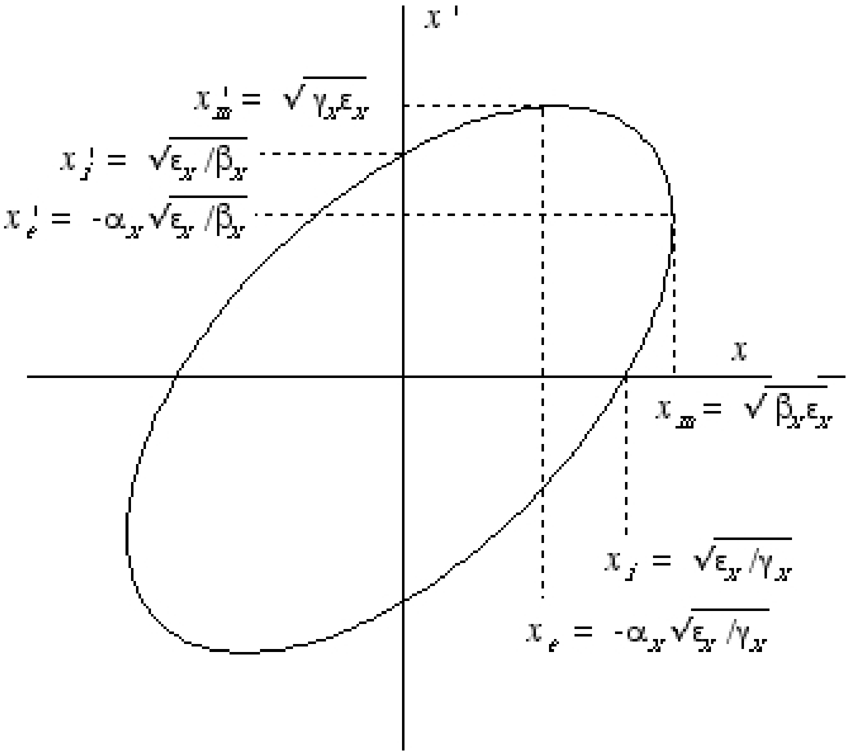


- **If you start with an collection of particles ("beam") with a set of initial amplitudes and phases inside of a given ellipse the ellipse will evolve ("down the beamline") and all particles will remain within that ellipse.**

⇒ Representation Provides a Useful Way to Describe a Beam
(at least for linear optics)

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

Relation Between σ-Matrix (Semi-Axis) Parameters and α-β (Twiss) Parameters



$$\beta_x \varepsilon_x \equiv \sigma_{11} = \langle x^2 \rangle = (x_{max})^2$$

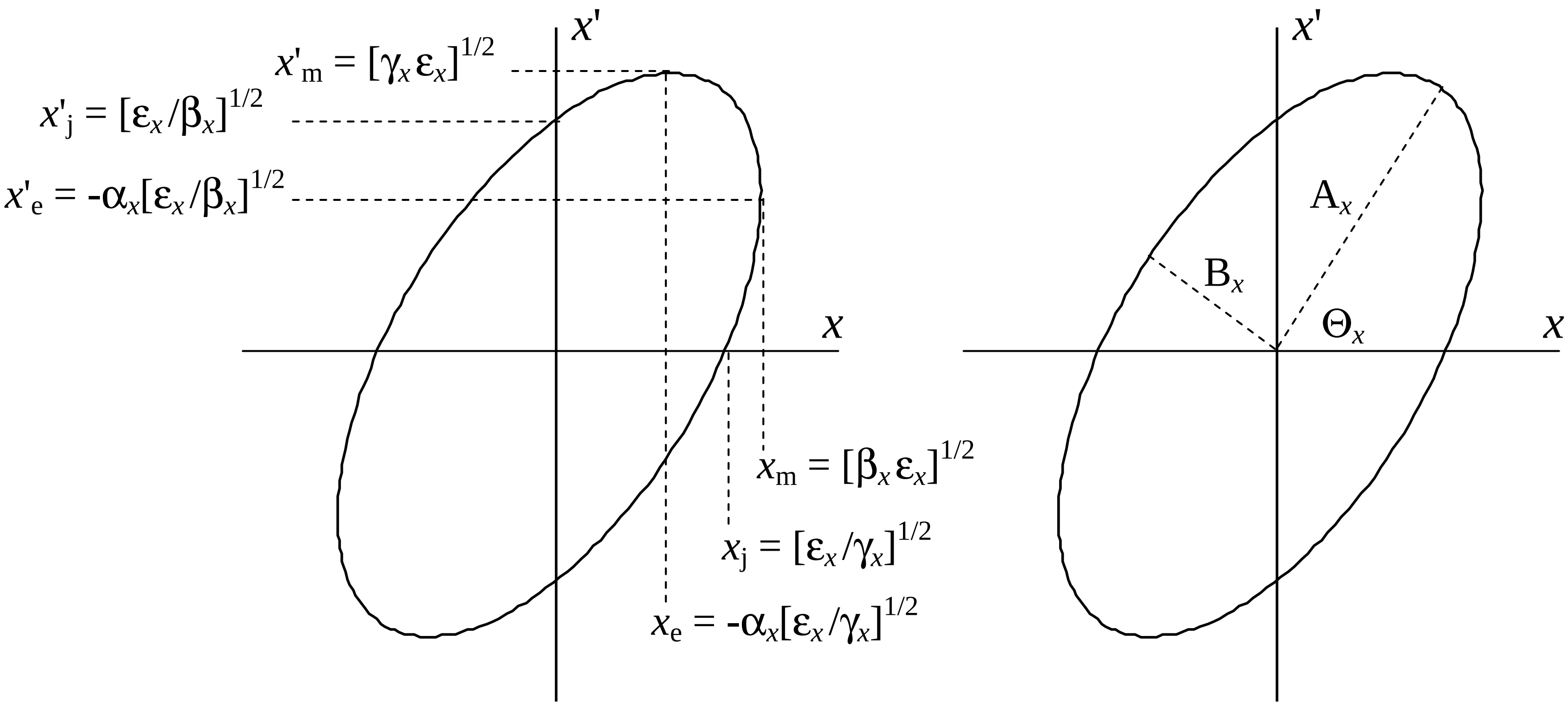
$$\gamma_x \varepsilon_x \equiv \sigma_{22} = \langle x'^2 \rangle = (x'_{max})^2$$

$$\alpha_x \varepsilon_x \equiv \sigma_{12} = \langle xx' \rangle = r_{12}\,[\sigma_{11}\sigma_{22}]^{1/2} \qquad \text{or } \alpha_x = -1 / [1 - r_{12}{}^2]^{1/2}$$

$$r_{12} \equiv \sigma_{12}/[\sigma_{11}\sigma_{22}]^{1/2} = r_{21} \equiv \sigma_{21}/[\sigma_{11}\sigma_{22}]^{1/2}$$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

Twiss Representation Geometric Parameterization

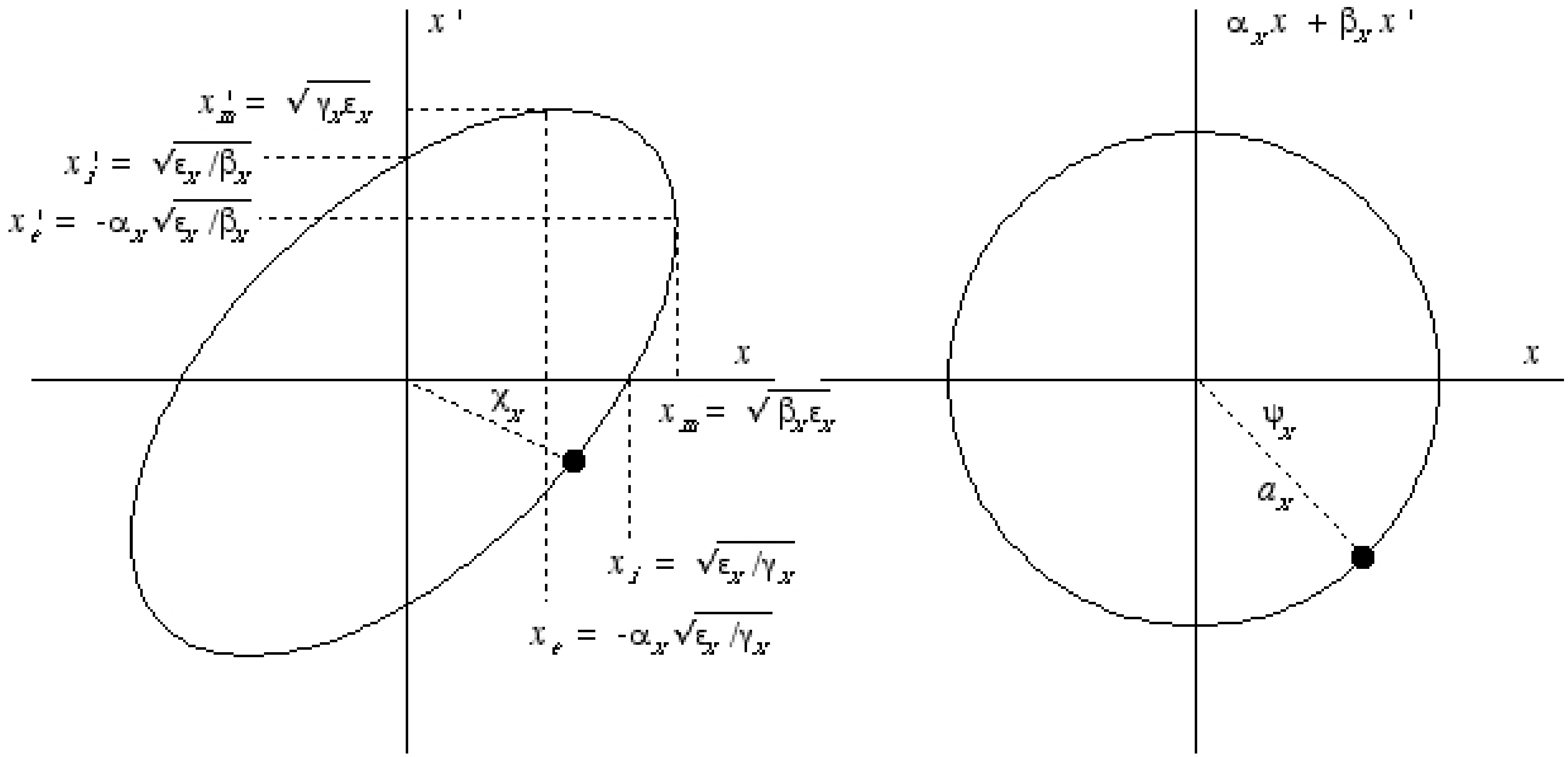


$$\gamma_x / \varepsilon_x = (\cos\Theta_x / A_x)^2 + (\sin\Theta_x / B_x)^2$$

$$\beta_x / \varepsilon_x = (\sin\Theta_x / A_x)^2 + (\cos\Theta_x / B_x)^2$$

$$\alpha_x / \varepsilon_x = \cos\Theta_x \sin\Theta_x \; [(1 / B_x)^2 - (1 / A_x)^2]$$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)
Note: Can Transform Ellipse into a Circle ⇒ Beam Comparisons (TRACE 3-D)



$$\beta_x \varepsilon_x = (a_x)^2$$

$$\tan\chi_x = (\tan\psi_x - \alpha_x) / \beta_x$$

alternatively: $\cos\psi_x = \cos\chi_x / [(\cos\chi_x)^2 + (\alpha_x \cos\chi_x - \beta_x \sin\chi_x)^2]^{1/2}$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

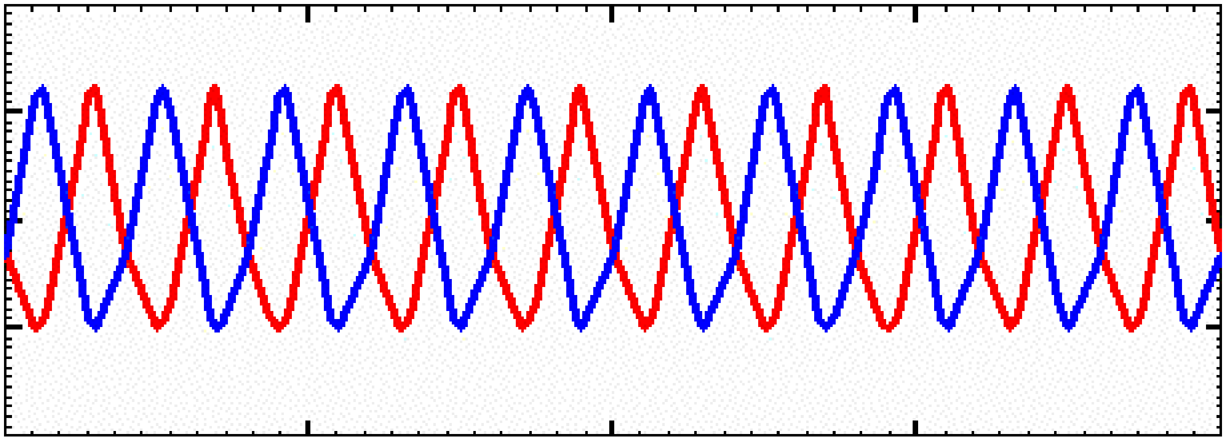
- **The parameters α_x, β_x and γ_x are often called "Twiss Parameters"**

  ⇒ I (try to) use the term Twiss Parameters when describing beam properties

- **The α_x, β_x and γ_x are also called the "Courant-Snyder Parameters"**

  ⇒ I use Courant-Snyder Parameters when describing machine properties

- **For a periodic (stable) system, the functions $\alpha_x(s)$ and $\beta_x(s)$ will be periodic**



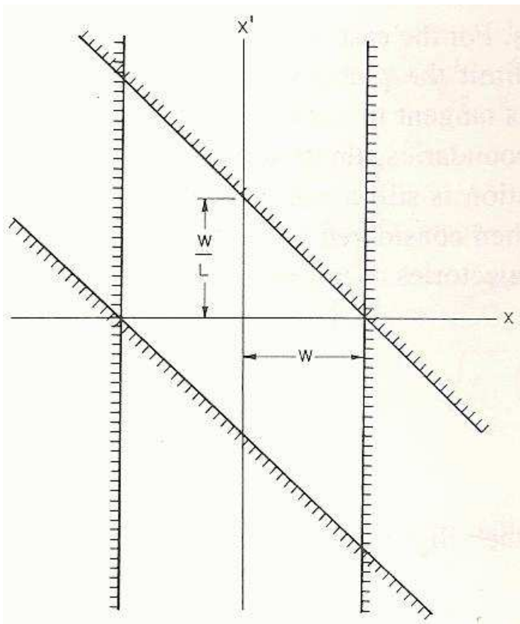
Example of $\beta_x(s)$ and $\beta_y(s)$ for a storage ring

- **The only beam that will survive many passes through such a system (i.e. many turns around the ring) is called the "matched beam"**

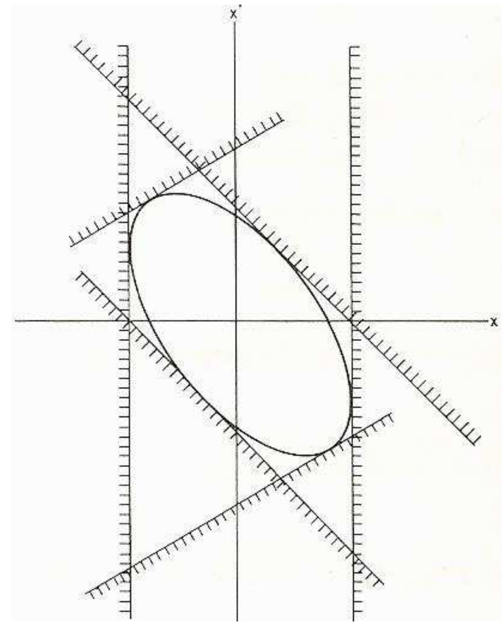
  ⇒ The "matched beam" requirements are machine properties

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Two slits will define an "acceptance" phase space for a beam**
- **Slit-defined "acceptance" phase space also specifies an enclosing ellipse**



Phase space accepted by two slits, each of width W, separated by L.



Phase space accepted by three slits, with focusing elements in between.

[Figures from page 101 of *The Optics of Charged Particle Beams* by David C. Carey, Harwood Academic Publishers (1987).]

- **An inscribed "acceptance" ellipse can be used to describe accepted beam**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

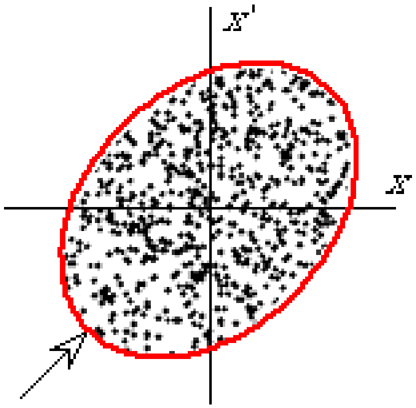
Emittances: RMS, Boundary, Normalized, Unnormalized, …

⇒ **The value of ε_x, which is a measure of the ellipse area, is the x-emittance.**

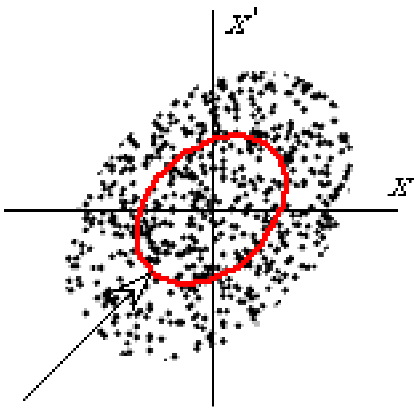
One convention defines ε_x as the area of the ellipse divided by π, and written as

$$\varepsilon_x = [\, \langle x^2 \rangle \langle x'^2 \rangle - \langle x'x \rangle^2]^{1/2} = 0.1\ \pi\text{-mm-mrad} \qquad (\pi \text{ in the units})$$

- **Emittance Definitions & Usage are Not Uniform**
- **Laboratory Emittance = "unnormalized" emittance (assumption so far)**
- **Emittance ellipse may describe all, or only a part, of the beam**



Enclosing Ellipse:
"boundary" emittance



Partially Enclosing Ellipse:
"RMS" emittance, "50% contour" emittance, ...

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

Emittances: RMS, Boundary, Normalized, Unnormalized, …

$\varepsilon_x = [\ \langle x^2\rangle \langle x'^2\rangle - \langle x'x\rangle^2]^{1/2}$ π-mm-mrad (π in the units)

- **Laboratory emittance (i.e. "unnormalized") not preserved with acceleration**

$$\varepsilon_x(E_b) = \varepsilon_x(E_a)\ [\beta_a\gamma_a] / [\beta_b\gamma_b]$$

⇒ **These $\beta_a\gamma_a$ and $\beta_b\gamma_b$ are relativistic velocity and energy parameters, not Twiss!**

- **Normalized emittance, $\varepsilon_{x,n}$, is preserved with acceleration**

$$\varepsilon_{x,n} = [\beta_a\gamma_a]\ \varepsilon_x(E_a)$$

- **Laboratory emittance scales with square root of kinetic energy** $E_a \rightarrow E_b$

$$\varepsilon_x(E_b) = \varepsilon_x(E_a)\ [\beta_a\gamma_a] / [\beta_b\gamma_b] \approx \varepsilon_x(E_a)\ [v_a/v_b] \approx \varepsilon_x(E_a)\ [E_a / E_b]^{1/2}$$

- **Reported emittance may also need to be scaled with mass** $m_a \rightarrow m_b$

$$\varepsilon_x(m_b) = \varepsilon_x(m_a)\ [\beta_a\gamma_a] / [\beta_b\gamma_b] \approx \varepsilon_x(m_a)\ [E_a / E_b]^{1/2}\ [m_b / m_a]^{1/2}$$

[Often emittances from ion source are different for different mass particles.]

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

Emittances: RMS, Boundary, Normalized, Unnormalized, …

- **Example:** KACST ion injector design acceptance:
- **Acceptance $\varepsilon_{acceptance}$ of beamline for 30 kV extraction is:**

$$\varepsilon_{acceptance} = 120\ \pi\text{-mm-mrad}$$

[from Table 1 of "A Versatile Ion Injector at KACST," M. O. A. El Ghazaly, S. A. Behery, A. A. Almuqhim. A. I. Papish and C. P. Welsch, AIP Conference Proceedings 1370, 272-277 (2011).]

- **Normalized or Un-Normalized? If We ("I") *Assume* Normalized by $\beta\gamma$:**
- **Extraction energy will be $E_a = 30\times q$ keV where q is the charge state of the ion**
- **The β,γ values for a (30 keV) proton ($q = 1$) are $\beta_a = 0.007997$ and $\gamma_a = 1.000032$**
- **The equivalent laboratory emittance to use in calculations is thus:**

$$\varepsilon_{lab}(30\text{ keV } H^+) = \varepsilon_{acceptance} / (\beta_a\gamma_a) = 120\ \pi\text{-mm-mrad} / 0.007997 = 15005.\ \pi\text{-mm-mrad}$$

- **For 30 kV charge 2 ($q = 2$) heavy (1500 amu) ion the emittance for calculations:**

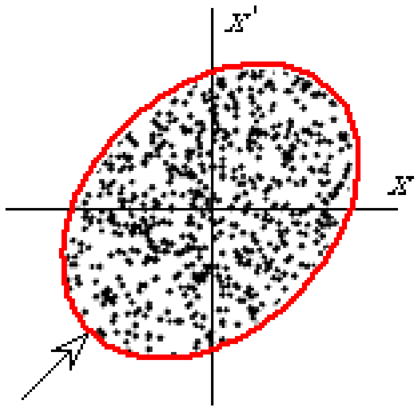
$$\varepsilon_{lab}(60\text{ keV 1500 amu}) = 15005\ \pi\text{-mm-mrad}\ [(30.0/60.0)^{1/2}(1500/1.007276)^{1/2}]$$
$$= 15005\times(0.7071\times38.59)\ \pi\text{-mm-mrad} = 409442\ \pi\text{-mm-mrad}$$

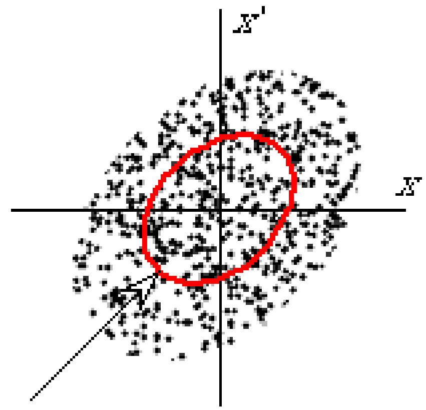
[Using $\beta_b = 0.000293$ and $\gamma_b = 1.000000$ the direct calculation gives same result:
$\varepsilon_{lab}(60\text{ keV 1500 amu}) = \varepsilon_{acceptance} / (\beta_b\gamma_b) = 120/0.000293\ \pi\text{-mm-mrad} = 409556\ \pi\text{-mm-mrad}$**]**

$\Rightarrow$ **These laboratory emittances seem "large" $\Rightarrow$ *Assumption* probably wrong.**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Virtually All Optics Codes (including TRACE 3-D) Use Laboratory Emittance**
- **TRACE 3-D Uses an "Equivalent Uniform Beam" Model of A Beam**





Enclosing Ellipse Gives Boundary "bnd" Emittance ≡ bnd Emittance = 5 × RMS Emittance For a Bunched (3-D) Beam

Enclosing Ellipse Gives bnd Emittance = 4 × RMS Emittance For Continuous (2-D or DC) Beam

- **PBO Lab Uses the TRACE 3-D Bunched Beam Convention:** $\varepsilon_{bnd} = 5\varepsilon_{rms}$
- **Beam "Sizes" $(5)^{1/2}$ × RMS Sizes:** $<x^2> = (x_{max})^2$ where $x_{max} = (2.2360...) \times x_{rms}$

3. Equations of Motion: Drifts, Quads, Bends

But how do we get from
$\underline{F} = m\underline{a}$
to the matrix formalism?

⇒ **Equations of Motion**

3. Equations of Motion: Drifts, Quads, Bends

- **Classical mechanics, Newton's 2nd Law:**

  $\mathbf{F} = d\mathbf{p}/dt$ **(F, p 3-vectors)**

- **Relativistically correct, with proper interpretation of F and p (need a 4-vector)**

- **Spatial components:**

  $F_x = dp_x/dt$ with $p_x = \beta_x \gamma mc$

  $F_y = dp_y/dt$ with $p_y = \beta_y \gamma mc$

  $F_z = dp_z/dt$ with $p_z = \beta_z \gamma mc$

- **4th component (energy W):**

$\mathbf{F} \cdot \mathbf{v} = dW/dt$ with $W^2 = p^2c^2 + m^2c^4$

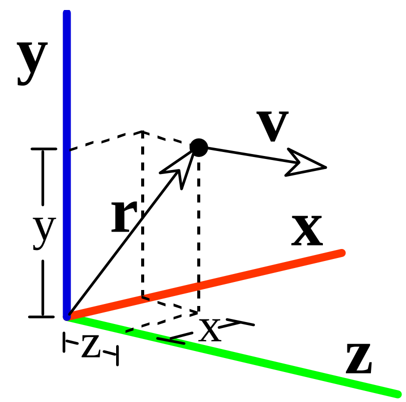
- **More elegant formulation uses Hamiltonian mechanics (not discussed further here)**

Equations of Motion F = ma Version

This section of the lecture uses:

BoldFont for 3-Vectors

Plain Font for Scalars



The z coordinate is often denoted by *l*

***l* = path length difference**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion (con't)

- **For cases where the Reference Trajectory is straight, and there is no acceleration (i.e. all magnetic elements except bends), the equations of motion in TRACE 3-D coordinates can be derived using the relation:**

$$d/dt = (ds/dt)\, d/ds \equiv c\beta_s\, d/ds, \qquad \text{e.g. } v_x \equiv dx/dt = c\beta_s dx/ds \equiv c\beta_s x'$$

- **Transverse motion (**x **and** y**):**

$$F_x = dp_x/dt = c\beta_s\, d(\beta_x\gamma mc)/ds = c\beta_s\, d(x'\beta_s\gamma mc)/ds = c\beta_s^2\gamma mc\, dx'/ds$$
$$F_y = dp_y/dt = c\beta_s\, d(\beta_y\gamma mc)/ds = c\beta_s\, d(y'\beta_s\gamma mc)/ds = c\beta_s^2\gamma mc\, dy'/ds$$
$$\text{where: } x' = dx/ds, \quad y' = dy/ds$$

- **Convenient to write these in the form:**

$$dx/ds = x' \qquad dx'/ds = [F_x / p_s]\, 1/(c\beta_s)\, (\gamma_s/\gamma)$$
$$dy/ds = y' \qquad dy'/ds = [F_y / p_s]\, 1/(c\beta_s)\, (\gamma_s/\gamma)$$

- **Use the Lorentz force to get the forces** F_x , F_y **for particular fields**

For a force free region (e.g. drift space) $F_x = F_y = 0$, hence $dx'/ds = dy'/ds = 0$

So that $dx/ds = x' = \text{constant}_x$, and $dy/ds = y' = \text{constant}_y$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion (con't)

- **Longitudinal motion in TRACE 3-D coordinates (l and δ), when the Reference Trajectory is straight and there is no acceleration (i.e. all magnetic elements except bends), is simple but non-trivial**
- **For magnetic fields, where $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$, then $\mathbf{F} \cdot \mathbf{v} = 0$, and $dW/dt = 0$**

$$dW/dt = c\beta_s\, dW/ds = c\beta_s\, d(\gamma mc^2)/ds = mc^3\beta_s\, d\gamma/ds = 0$$

If $d\gamma/ds = 0$ **, then** $d\beta/ds = 0$ **also, and likewise** $d(\beta\gamma)/ds = 0$

- **So with no acceleration, then for the Reference Trajectory variables**

$$d(\beta_s\gamma_s)/ds = 0$$

and since $\delta = \left[(\beta\gamma)/(\beta_s\gamma_s)\right] - 1$ **one then has:**

$$d\delta/ds = 0 \qquad (\Rightarrow \text{Conservation of Energy, Bends too})$$

- **More general case (e.g. with acceleration) one can show that**

$$\frac{d\delta}{dz_s} = \frac{1}{(1+\delta)}\frac{\beta_z\gamma}{\beta_s\gamma_s}\frac{1}{c\beta_s p_s}\left[F_z + \left(\frac{v_x}{v_z}\right)F_x + \left(\frac{v_y}{v_z}\right)F_y\right] - \frac{(1+\delta)}{\beta_s\gamma_s}\frac{d(\beta_s\gamma_s)}{dz_s}$$

- **What about the longitudinal coordinate l ?**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion (con't)

- l **is the projected (on z direction) path length difference**

  $dl/dt = c\beta_s \, dl/ds$, **but** $dl/dt = v_z - v_s = c(\beta_z - \beta_s)$, **hence**

  $$dl/ds = (\beta_z/\beta_s) - 1$$

- **Longitudinal velocity** β_z **(not conserved) in terms of other variables (**x', y', δ**)**

  $$\beta_z = \left\{(\beta)^2 - (\beta_x)^2 - (\beta_y)^2\right\}^{1/2} = \left\{(\beta)^2 - (x'\beta_s)^2 - (y'\beta_s)^2\right\}^{1/2}$$

  it can be shown that $\beta = \beta_s (1 + \delta) / \left[1 + \delta(2+\delta)(\beta_s)^2\right]^{1/2}$, **hence**

  $$\beta_z = \beta_s \left\{\left((1 + \delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - (x')^2 - (y')^2\right\}^{1/2}$$ **, so finally:**

  $$dl/ds = \left\{\left((1 + \delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1$$

- **Note that** $dl'/ds \neq 0$ **, where** $l' = dl/ds$**, but has an "apparent" "force"** F_l **:**

  $$F_l = (d/ds)\left\{\left((1 + \delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2}$$

- **Longitudinal coordinate** l **important for radiofrequency (RF) components**

[• **Not all codes use the (TRACE 3-D, TRANSPORT) longitudinal coordinate** l]

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Drift

For (non dipole) magnetic systems without acceleration, the equations of motion in TRACE 3-D variables x, x', y, y', l, δ **are:**

$$dx/ds = x' \qquad dx'/ds = [F_x / p_s]\, 1/(c\beta_s) \left(1/\left\{1 + \delta(2+\delta)(\beta_s)^2\right\}^{1/2}\right)$$

$$dy/ds = y' \qquad dy'/ds = [F_y / p_s]\, 1/(c\beta_s) \left(1/\left\{1 + \delta(2+\delta)(\beta_s)^2\right\}^{1/2}\right)$$

$$dl/ds = \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1 \qquad d\delta/ds = 0$$

Return to the simplest example: Drift (field free region):

$$dx/ds = x' \qquad dx'/ds = 0$$

$$dy/ds = y' \qquad dy'/ds = 0$$

$$dl/ds = \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1 \qquad d\delta/ds = 0$$

Integrate ds **from** s_a **to** s_b**, with** $s_b - s_a = L$**, the length of the drift, the solutions are:**

$$x_b = x_a + x'_a L \qquad x'_b = x'_a$$

$$y_b = y_a + y'_a L \qquad y'_b = y'_a$$

$$l_b = l_a + \left\{\left((1+\delta_o)^2 / \left[1 + \delta_o (2+\delta_o)(\beta_s)^2\right]\right) - x'^2_a - y'^2_a\right\}^{1/2} L - L \qquad \delta_b = \delta_a \equiv \delta_o$$

⇒ **Drift is inherently nonlinear for the longitudinal coordinate**

$$l_b \cong l_a + (\delta_o/\gamma_s^2) L + O(\delta_o^2, x'^2_a, y'^2_a) + \text{higher order terms}$$

[• **Although codes may not use the (TRACE 3-D, TRANSPORT) longitudinal coordinate** l**, their <u>equivalent</u> longitudinal coordinates are still nonlinear**]

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Drift (con't)

- **Solution for Drift is in the form of matrix equation, taking an initial vector** $[q_{i\,a}] = (x_o, x'_o, y_o, y'_o, l_o, \delta_o)$ **to a final vector** $[q_{i\,b}] = (x, x', y, y', l, \delta)$**, where the** R-Matrix **equation,** $q_{i\,b} = [R]\, q_{i\,a}$**, is obtained using only** $l = (\delta_o/\gamma_s^2)\,L + l_o$.
- **Useful to break (6-by-6)** R-Matrix **into a set of 9 (2-by-2) submatrices:**

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{bmatrix} = R \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} [R_{xx}] & [R_{xy}] & [R_{xz}] \\ [R_{yx}] & [R_{yy}] & [R_{yz}] \\ [R_{zx}] & [R_{zy}] & [R_{zz}] \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix}$$

- **For a drift, most submatrices are zero, only three are non-zero:**

$$R_{xx} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & s/\gamma_s^2 \\ 0 & 1 \end{bmatrix}$$

- **For a drift of length** L**, then the** R-Matrix **above is evaluated for** $s = L$
- **Easy to show that, for two drifts of lengths** L_1 **and** L_2**, the multiplication of the two** R-Matrices **is simply a** R-Matrix **for length** $L = L_1 + L_2$

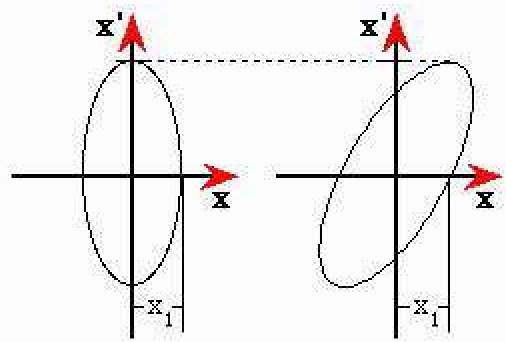
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

R-Matrix Example - Drift

- **250 keV protons (β = 0.023080 and γ = 1.000266)**
- **Drift Length of 2 Meter**

The Drift Piece is an *Approximate First Order* Optics Element

The non-trivial sub matrices for the Drift Piece are determined by the value of the Effective Drift Length L in the Piece Window, and the relativistic energy γ. These submatrices are:



$$[R_{xx}] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 2.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$$

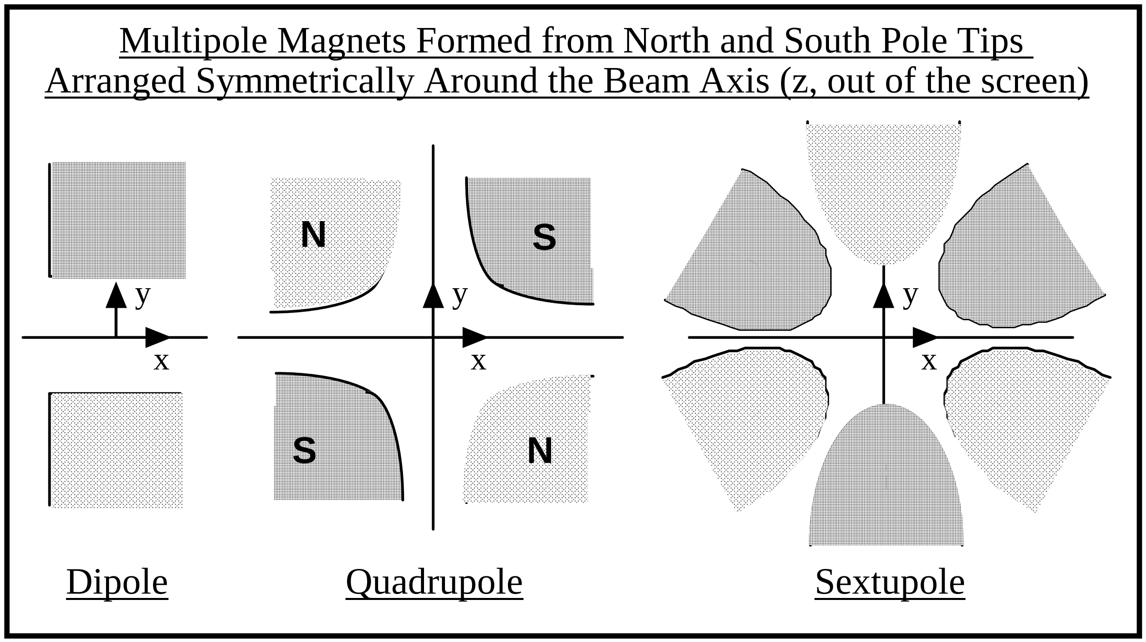
$$[R_{yy}] = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 2.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$$

$$[R_{zz}] = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} = \begin{bmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 1.9989 \\ 0.0000 & 1.0000 \end{bmatrix}$$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole

- **Magnetic quadrupole is one example of magnetic cylindrical multipole field**



- **Magnetic Quadrupole Field: inside:** $\mathbf{B} = (B_o/a)\,[(r\sin\theta)\,\mathbf{x} + (r\cos\theta)\,\mathbf{y}]$ $(0<z<L)$
  (in absence of fringe fields) outside: $\mathbf{B} = 0$ $(z<0 \text{ or } z>L)$
- **Lorentz Force Gives** F_x **and** F_y**:** $\mathbf{F} = (q)\,[\mathbf{E} + \mathbf{v}\times\mathbf{B}] = (q)\,\mathbf{v}\times\mathbf{B}$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

Key Parameters

For Optics:

Pole Tip Field, B_o

Bore Radius, a $(=r_1)$

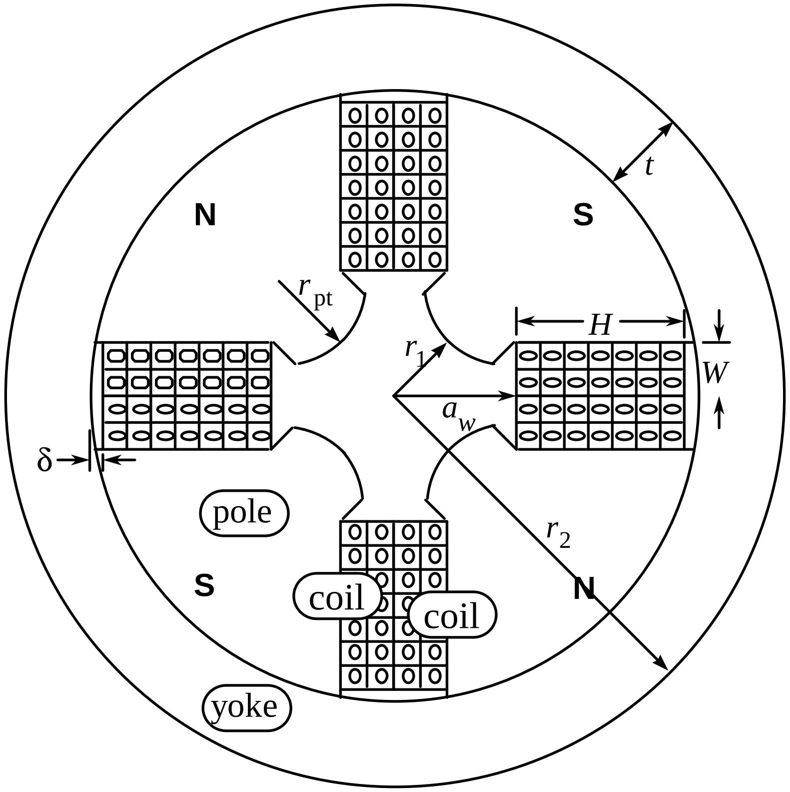
Effective Length, L

For Engineering:

No. of Turns/Coil, n

Current in Coil, I

Typical Design of an Electromagnetic Quadrupole Showing Key Parameters Defining the Pole-Tip Geometry and Field Coils.



3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

To Relate Engineering & Optics Parameters: Use Ampere's Law

Separate Integral into Four Parts

$$\oint \boldsymbol{H} \bullet \boldsymbol{ds} = I(1) + I(2) + I(3) + I(4)$$

$$\oint \boldsymbol{H} \bullet \boldsymbol{ds} = nI$$

I(1) **Main Contribution from Bore Radial Integral**

$$I(1) = \int_0^{r_1} H_{pt} dr = \int_0^{r_1} (B_o/\mu_o)(r/r_1) dr = B_o r_1/(2\mu_o)$$

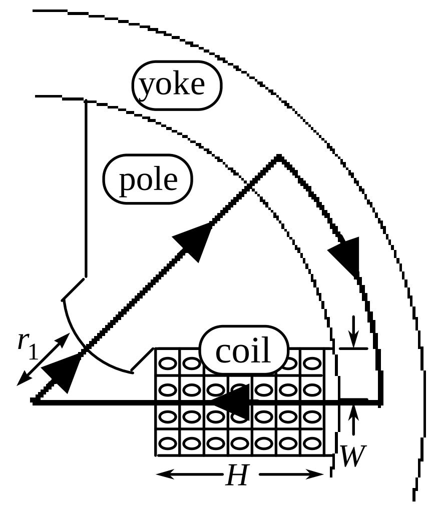
I(2)+ I(3) **Contributions of Pole and Yoke Small ($\mu >> 1$)**

$$I(2) = \int_{r_1}^{r_{int}} \boldsymbol{H}_{pole} \bullet \boldsymbol{dr} = (1/\mu_{pole}) \int_{r_1}^{r_{int}} \boldsymbol{B}_{pole} \bullet \boldsymbol{dr}$$

$$I(3) = (r_{int}) \int_0^{\pi/4} \boldsymbol{H}_{yoke} \bullet \boldsymbol{d\Theta} = (1/\mu_{yoke})(r_{int}) \int_0^{\pi/4} \boldsymbol{B}_{yoke} \bullet \boldsymbol{d\Theta}$$

I(4) **Contribution Vanishes ($\boldsymbol{H} \perp \boldsymbol{dr}$)**

$$I(4) = \int_{r_{int}}^{0} \boldsymbol{H} \bullet \boldsymbol{dr} = 0 \quad \Rightarrow \quad B_O = 2\mu_O \, nI / r_1$$



Integration Path

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

- **Magnetic Field:** $\mathbf{B} = (B_o/a)\,[(r\sin\theta)\,\mathbf{x} + (r\cos\theta)\,\mathbf{y}] = (B_o/a)\,[(y)\,\mathbf{x} + (x)\,\mathbf{y}]$, **or**

$$B_x = (B_o/a)\,y = B'\,y \qquad B_y = (B_o/a)\,x = B'\,x$$

- **Force Components:** $F_x = -\,(qc)\,\beta_z B_y$ **and** $F_y = (qc)\,\beta_z B_x$ **, or**

$$F_x = -\,(qcB')\,\beta_z x \qquad F_y = (qcB')\,\beta_z y$$

- **Equations of Motion**

$$dx/ds = x' \qquad dx'/ds = -\,[(qc)\,\beta_z B'/p_s]\,1/(c\beta_s)\,(\gamma_s/\gamma)\,x = -\,[qB'/p_s](\beta_z/\beta_s)(\gamma_s/\gamma)\,x$$

$$dy/ds = y' \qquad dy'/ds = +\,[(qc)\,\beta_z B'/p_s]\,1/(c\beta_s)\,(\gamma_s/\gamma)\,y = +\,[qB'/p_s](\beta_z/\beta_s)(\gamma_s/\gamma)\,y$$

$dl/ds = (\beta_z/\beta_s)-1 \qquad d\delta/ds = 0$ **(longitudinal same as a drift)**

- **Expand non-linear terms** $(\beta_z/\beta_s) \cong 1 + (\delta_o/\gamma_s^2) + \ldots$ **,** $(\beta_z/\beta_s)(\gamma_s/\gamma) \cong 1 - \delta_o + \ldots$

$dx/ds = x' \qquad dx'/ds = -\,[qB'/p_s]\,x = -\,[K_1]\,x$ — K_1 **is Quad Coefficient:**

$dy/ds = y' \qquad dy'/ds = +\,[qB'/p_s]\,y = +\,[K_1]\,x$ — $K_1 = [qB'/p_s] = (q/p_s)\,[B_o/a]$

$dl/ds = \delta_o/\gamma_s^2 \qquad d\delta/ds = 0$ **(longitudinal same as a drift)**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

- **Transverse motion**

$dx/ds = x'$ $\quad dx'/ds = dx^2/ds^2 = -[K_1]\,x$ **or** $dx^2/ds^2 + [K_1]\,x = 0$

$dy/ds = y'$ $\quad dy'/ds = dy^2/ds^2 = +[K_1]\,x$ **or** $dy^2/ds^2 - [K_1]\,y = 0$

- **First order: simple harmonic type motion, solutions depend upon the sign of** K_1**. Useful to define** $k = |K_1|^{1/2}$**, then for** $K_1 > 0$**:**

$$x = x_o\cos(ks) + x'_o\sin(ks)/k \qquad x' = -kx_o\sin(ks) + x'_o\cos(ks)$$

$$y = y_o\cosh(ks) + y'_o\sinh(ks)/k \qquad y' = +ky_o\sinh(ks) + y'_o\cosh(ks)$$

$K_1 > 0$ **is an "x-focusing" quad, and x is typically taken to be horizontal direction**

- **For** $K_1 < 0$**, the trigonometric and hyperbolic functions are exchanged:**

$$x = x_o\cosh(ks) + x'_o\sinh(ks)/k \qquad x' = +kx_o\sinh(ks) + x'_o\cosh(ks)$$

$$y = y_o\cos(ks) + y'_o\sin(ks)/k \qquad y' = -ky_o\sin(ks) + y'_o\cos(ks)$$

- **One phase plane is focusing (trigonometric functions) and one phase plane is defocusing (hyberbolic functions)**

$\Rightarrow$ **Need at least 2 quads for focusing in both transverse directions**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

- **The Result is a Block Diagonal** R-Matrix**:**

$$R = \begin{bmatrix} R_{xx} & 0 & 0 \\ 0 & R_{yy} & 0 \\ 0 & 0 & R_{zz} \end{bmatrix}$$

- **For an x-focusing (**$K_1 > 0$**) quad, the three non-zero submatrices are:**

$$R_{xx} = \begin{bmatrix} \cos(ks) & \sin(ks)/k \\ -k\sin(ks) & \cos(ks) \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} \cosh(ks) & \sinh(ks)/k \\ k\sinh(ks) & \cosh(ks) \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & s/\gamma_s^2 \\ 0 & 1 \end{bmatrix}$$

- **For a quad of length** L**, then the** R-Matrix **above is evaluated for** $s = L$

- **A useful case is where** $kL << 0$**, but** k^2L **is *not* "small", then** R-Matrix **is:**

$$R_{xx} = \begin{bmatrix} 1 & 0 \\ -k^2L & 1 \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} 1 & 0 \\ +k^2L & 1 \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ **Thin Lens approximation with focal lengths** $f_x = 1/(k^2L)$ **and** $f_y = -1/(k^2L)$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

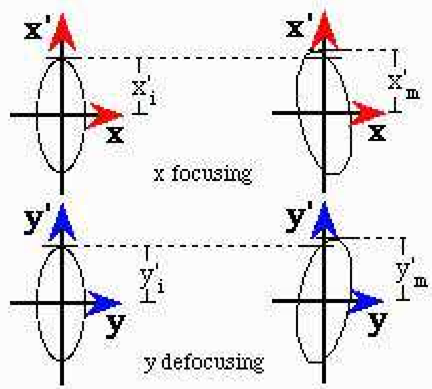
R-Matrix Example - Quadrupole 1: QL-100

- **250 keV protons ($\beta = 0.023080$ and $\gamma = 1.000266$)**
- **Length L of 6 cm, Aperture a of 1.7 cm, Pole Tip Field B_o of 0.034120 T**

(Gradient $B' = 20.0706$ T/m, Quadrupole Coefficient $K_1 = 277.78\ \text{m}^{-2}$, $k = 16.667\ \text{m}^{-1}$)

The Quadrupole is an *Approximate First Order* Optics Element

The non-trivial submatrices for the Quadrupole Piece are determined by the value of the Quadrupole Coefficient $K_1 = k^2$, **the** Effective Length L, **and the relativistic energy** γ. **The submatrices for a x-focusing quadrupole are:**



$$[\mathbf{R}_{xx}] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \cos(kL) & \sin(kL)/k \\ -k\sin(kL) & \cos(kL) \end{bmatrix} = \begin{bmatrix} 0.5403 & 0.0505 \\ -14.0246 & 0.5403 \end{bmatrix}$$

$$[\mathbf{R}_{yy}] = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} \cosh(kL) & \sinh(kL)/k \\ k\sinh(kL) & \cosh(kL) \end{bmatrix} = \begin{bmatrix} 1.5431 & 0.0705 \\ 19.5869 & 1.5431 \end{bmatrix}$$

$$[\mathbf{R}_{zz}] = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} = \begin{bmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.0600 \\ 0.0000 & 1.0000 \end{bmatrix}$$

$\Rightarrow$ **Thin Lens approximation gives focal lengths $f_x = -f_y = 0.06$ m**

$\Rightarrow$ **Thick Lens focal lengths $f_x = -1/R_{21} = 0.0713$ m and $f_y = -1/R_{43} = -0.0511$ m**

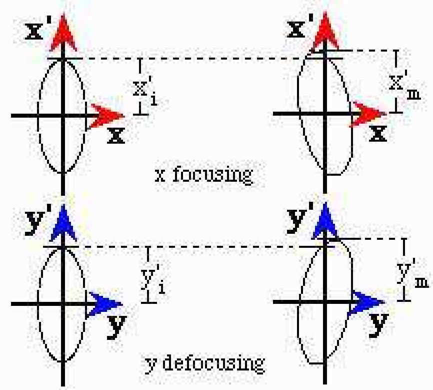
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

R-Matrix Example - Quadrupole 2: QL-100

- **10 keV Phosphorous (β = 0.000833 and γ = 1.000000)**
- **Length L of 6 cm, Aperture a of 1.7 cm, Pole Tip Field B_o of 0.037840 T (Gradient B'=22.2588 T/m, Quadrupole Coefficient K_1=277.79 m^{-2}, k=16.667 m^{-1})**

The Quadrupole is an *Approximate First Order* Optics Element

The non-trivial submatrices for the Quadrupole Piece are determined by the value of the Quadrupole Coefficient $K_1 = k^2$, the Effective Length L, and the relativistic energy γ. The submatrices for a x-focusing quadrupole are:



$$[R_{xx}] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \cos(kL) & \sin(kL)/k \\ -k\sin(kL) & \cos(kL) \end{bmatrix} = \begin{bmatrix} 0.5403 & 0.0505 \\ -14.0251 & 0.5403 \end{bmatrix}$$

$$[R_{yy}] = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} \cosh(kL) & \sinh(kL)/k \\ k\sinh(kL) & \cosh(kL) \end{bmatrix} = \begin{bmatrix} 1.5431 & 0.0705 \\ 19.5879 & 1.5431 \end{bmatrix}$$

$$[R_{zz}] = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} = \begin{bmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.0600 \\ 0.0000 & 1.0000 \end{bmatrix}$$

$\Rightarrow$ **Thin Lens approximation gives focal lengths $f_x = -f_y = 0.06$ m**

$\Rightarrow$ **Thick Lens focal lengths $f_x = -1/R_{21} = 0.0713$ m and $f_y = -1/R_{43} = -0.0511$ m**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Some ElectroMagnetic Quadrupole (EMQ) Formulas

- **Pole Tip Magnetic Field**

$$B_o = 2\mu_o \, nI / r_1$$

$$B_o(\mathrm{T}) = 8\pi \times 10^{-7} \, [nI(\mathrm{Amps})]/[r_1(\mathrm{m})]$$

- **Quadrupole Gradient**

$$B' = B_o / r_1 = 2\mu_o \, nI / (r_1)^2$$

$$B'(\mathrm{T/m}) = 8\pi \times 10^{-7} \, [nI(\mathrm{Amps})]/[r_1(\mathrm{m})]^2$$

- **Quadrupole Strength**

$$\kappa \equiv K_1 = B' / [B\rho]$$

$$\kappa(\mathrm{m}^{-2}) = 0.299792 [B'(\mathrm{T/m})]/[p(\mathrm{GeV})]$$

- **Effective Length**

$$L = [B'(0)]^{-1} \int B'(z) \, dz \sim (0.75 - 0.97) \times (\text{physical length w/coils})$$

- **Equivalent Thin Lens Focal Length**

$$f = 1 / [\kappa L] = [B\rho]/[B' L]$$

$$f(\mathrm{m}) = [p(\mathrm{GeV})]/[0.299792 \, L(\mathrm{m}) \, B'(\mathrm{T/m})]$$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - ElectroStatic Quadrupole

Key Parameters

For Optics:

Pole Voltage, V_o

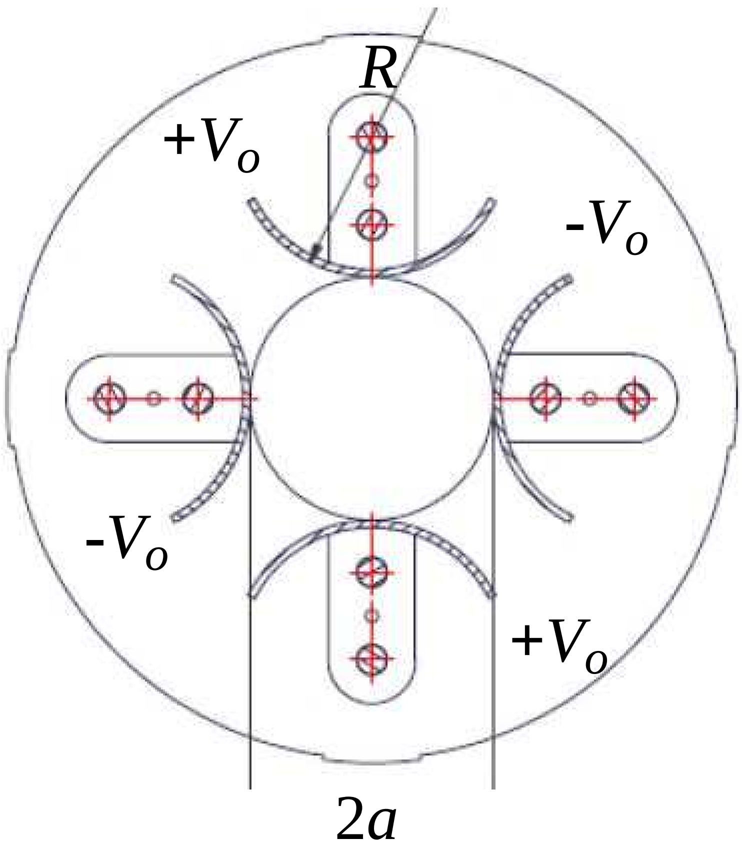
Bore Radius, a

Effective Length, L

For Engineering:

Electrode Radius, R

Power, Vacuum Considerations



- **Lorentz Force Gives** F_x **and** F_y**:** $\mathbf{F} = (q)\,[\mathbf{E} + \mathbf{v} \times \mathbf{B}] = (q)\,\mathbf{E}$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - ElectroStatic Quadrupole (con't)

- **Electric Field: E** = - $(2V_o/a^2)$ [(r cosθ) **x** - (r sinθ) **y**] = $(-2V_o/a^2)$ [(x) **x** - (y) **y**], **or**

$$E_x = (-2V_o/a^2)\, x = -\, G\, x \qquad E_y = -(-2V_o/a^2)\, y = +\, G\, y$$

- **Force Components:** $F_x = q(e)\, E_x$ **and** $F_y = q(e)\, E_y$ **, or**

$$F_x = -\,(2qV_o/a^2)\, x \qquad F_y = (2qV_o/a^2)\, y$$

- **Equations of Motion**

$$dx/ds = x' \qquad dx'/ds = -\,[(2qV_o/a^2)/p_s]\,(\gamma_s/\gamma)/(c\beta_s)\, x = -\,[qG/p_s]\,(\gamma_s/\gamma)/(c\beta_s)\, x$$

$$dy/ds = y' \qquad dy'/ds = +\,[(2qV_o/a^2)/p_s]\,(\gamma_s/\gamma)/(c\beta_s)\, y = +\,[qG/p_s]\,(\gamma_s/\gamma)/(c\beta_s)\, y$$

$dl/ds = (\beta_z/\beta_s)-1 \qquad d\delta/ds = 0$ **(longitudinal same as a drift)**

- **Expand non-linear terms** $(\gamma_s/\gamma) \cong 1 - \beta_s^2\delta_o + \ldots$ (here is a difference from magnetic)

$dx/ds = x' \qquad dx'/ds = -\,[qG/(p_s c\beta_s)]\, x = -\,[K_1]\, x$ **K_1 is Quad Coefficient:**

$dy/ds = y' \qquad dy'/ds = +\,[qG/p_s]\, y = +\,[K_1]\, x$ **$K_1 = [qG/(p_s c\beta_s)] = (q/p_s)\,[2V_o/(a^2 c\beta_s)]$**

$dl/ds = \delta_o/\gamma_s^2 \qquad d\delta/ds = 0$ **(longitudinal same as a drift)**

⇒ ***1st Order*** **Equations of Motion Same as Magnetic Quad with** $K_1 = (q/p_s)\,[2V_o/(a^2 v_s)]$

⇒ **Differences Between Electrostatic & Magnetic Quads Occur in Higher Orders**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Some ElectroStatic Quadrupole (ESQ) Formulas

- **ElectroStatic (ES) Quadrupole Strength**

$$K_1(p) = 2qV_0/(a^2\beta p) \ \ \& \ \ \beta = \beta(p_s,\delta)$$

- **ES Quadrupole Gradient**

$$G = E_o/a = 2qV_0/a^2$$
$$G\ (\text{Volts/m}^2) = 2\ [q(e)]\ [V_0(\text{Volts})]/[a(\text{m})]^2$$

- **ES Quadrupole Strength**

$$\kappa \equiv K_1 = (q/p_s)\ [2V_o/(a^2 c\beta_s)]$$

- **Effective Length**

$$L = [G(0)]^{-1} \int G(z)\ dz \ \sim \ \text{electrode physical length}$$

- **Equivalent Thin Lens Focal Length**

$$f = 1/[\kappa L] = [p_s a^2 c\beta_s]/[2qV_o L]$$

- **Equivalent Magnetic Quadrupole Gradient**

$$B' = 2qV_0/(a^2 c\beta_s)$$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

What about higher order optics?

- **Second-order (chromatic aberrations) are (relatively) straightforward**
- **Replace the Reference Momentum with the Actual Momentum (e.g. in the Quadrupole Coefficient K_1) and expand in δ**

$$K_1(p) = [qB'/p] = (p_s/p)\,[qB'/p_s]\,] \equiv (p_s/p)\,K_1(p_s)$$

$$p \equiv (1+\delta)\,p_s$$

$$\text{so:} \quad (p_s/p) \approx 1 - \delta + O(\delta^2)$$

- **Solution to second-order equations of motion found using the Green's function approach for solving differential equations (PBO Lab).**
- **Third-order requires considerably more work**

  - Intrinsic third-order is independent of fringe-fields

  - Fringe-field third-order require at least four integrals (Matsuda & Wollnik)

- **TRACE 3-D incorporates fringe-fields by stepping through (integrating)**

  - Added fringe-field third-order terms to TRANSPORT, TURTLE

  - Added Electrostatic Quadrupole to TRANSPORT, TURTLE:
  First-, Second- and Third-order: $K_1(p) = 2qV_0/(a^2\beta p)$ & $\beta = \beta(p_s,\delta)$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Bend

- **Reference Trajectory follows an arc → curvilinear coordinates used**
- **For idealized Sector Dipole, 5 of the 9 submatrices are non-zero:**

$$R_{xx} = \begin{bmatrix} \cos(hs) & \sin(hs)/h \\ -h\sin(hs) & \cos(hs) \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & s/\gamma_s^2 \\ 0 & 1 \end{bmatrix}$$

$$R_{xz} = \begin{bmatrix} 0 & [1-\cos(hs)]/h \\ 0 & \sin(hs) \end{bmatrix} \qquad R_{zx} = \begin{bmatrix} -\sin(hs) & -[1-\cos(hs)]/h \\ 0 & 0 \end{bmatrix}$$

where $h = 1/\rho$ **and** ρ **is the bend radius**

- **Dispersion & compaction are introduced by the** R_{xz} **and** R_{zx} **submatrices**
- **Non-Idealized Sector Dipoles are More Common**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Bend (con't)

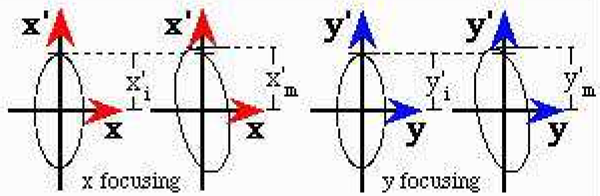
- **Non-Idealized Sector Dipole Effects that Impact First-Order Optics:**
  - **Fringe Fields**
  - **Pole Face Rotations**
  - **Pole Shoe Rotations**
- **First Order Fringe Field Effects and Pole Face Rotations are Often Referred to a "Edge Focusing"**
  - **Can be Modeled with Thin Lens at Entrance/Exit**
- **Pole Shoe Rotations Result in Non-Uniform B Field**
  - **First Order Effect is Radial Derivative of B Field**
  - **Often Referred to as a Gradient Bend**
  - **Can be Modeled with a Quadrupole Field added to Dipole**
- **Other Deviations Can also Useful**
  - **Curved Pole Faces, Higher-Order Combined Function Bends, ...**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

R-Matrix Example - Mass Separator Bend

- **10 keV Phosphorous (β = 0.000833 and γ = 1.000000)**
- **90° Gradient Bend L of 0.785398 m, Field Index n of 1/2, Pole Tip Field B_o of 0.160 T, bend radius ρ of 50 cm**

The Bend is an *Approximate First Order* Optics Element



Gradient bend properties are determined by the Field Gradient Index n, Central Trajectory Radius ρ_o(= 1/h) **and** Central Trajectory Length L, **with** $k_x^2=(1-n)h^2$, $k_y^2=nh^2$:

$$[R_{xx}] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \cos(k_x L) & \sin(k_x L)/k_x \\ -k_x \sin(k_x L) & \cos(k_x L) \end{bmatrix} = \begin{bmatrix} 0.4440 & 0.6336 \\ -1.2672 & 0.4440 \end{bmatrix}$$

$$[R_{xz}] = \begin{bmatrix} R_{15} & R_{16} \\ R_{25} & R_{26} \end{bmatrix} = \begin{bmatrix} 0 & h\,[1-\cos(k_x L)]/k_x^2 \\ 0 & h \sin(k_x L)/k_x \end{bmatrix} = \begin{bmatrix} 0.0000 & 0.5560 \\ 0.0000 & 1.2672 \end{bmatrix}$$

$$[R_{yy}] = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} \cos(k_y L) & \sin(k_y L)/k_y \\ -k_y \sin(k_y L) & \cos(k_y L) \end{bmatrix} = \begin{bmatrix} 0.4440 & 0.6336 \\ -1.2672 & 0.4440 \end{bmatrix}$$

$$[R_{zx}] = \begin{bmatrix} R_{51} & R_{52} \\ R_{61} & R_{62} \end{bmatrix} = \begin{bmatrix} -h \sin(k_x L)/k_x & -h\,[1-\cos(k_x L)]/k_x^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1.2672 & -0.5560 \\ 0.0000 & 0.0000 \end{bmatrix}$$

$$[R_{zz}] = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} = \begin{bmatrix} 1 & -[k_x L - \sin(k_x L)]/k_x + L/\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.4818 \\ 0.0000 & 1.0000 \end{bmatrix}$$