Overview of Particle Beam Optics Utilized in the Matrix, Envelope, and Tracking Codes: TRANSPORT, TRACE 3-D, and TURTLE

George H. Gillespie

G. H. Gillespie Associates, Inc.
P. O. Box 2961
Del Mar, California 92014, U.S.A.

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Presentation Outline - Part I

Overview of Particle Beam Optics Utilized in the Matrix, Envelope, and Tracking Codes: TRANSPORT, TRACE 3-D, and TURTLE

1. **Basic Matrix Premise, Coordinates, Linear / Nonlinear Particle Optics, …**
2. **Describing a Beam - Phase Space, Semi-Axes & Twiss Representations**

   ⇒ **Break**

3. **Equations of Motion: Drifts, Quads, Bends - Individual Particle Motion**

   ⇒ **Break**

4. **Introduction to the Beam Optics of TRANSPORT**
5. **Introduction to the Beam Optics of TRACE 3-D**
6. **Introduction to the Beam Optics of TURTLE**
7. **Summary**

Part II ⇒ **Will use the PBO Lab software in the class to illustrate concepts**

Presentation Outline - Part II

Overview of the PBO Lab Software with Examples and Sample Problems using TRANSPORT, TRACE 3-D, and TURTLE

8. Overview of the Particle Beam Optics Laboratory (PBO Lab)

Graphical Beamline Construction Kit
Interactive Tutorials
Built In First-Order Tools (Focusing & Bending)

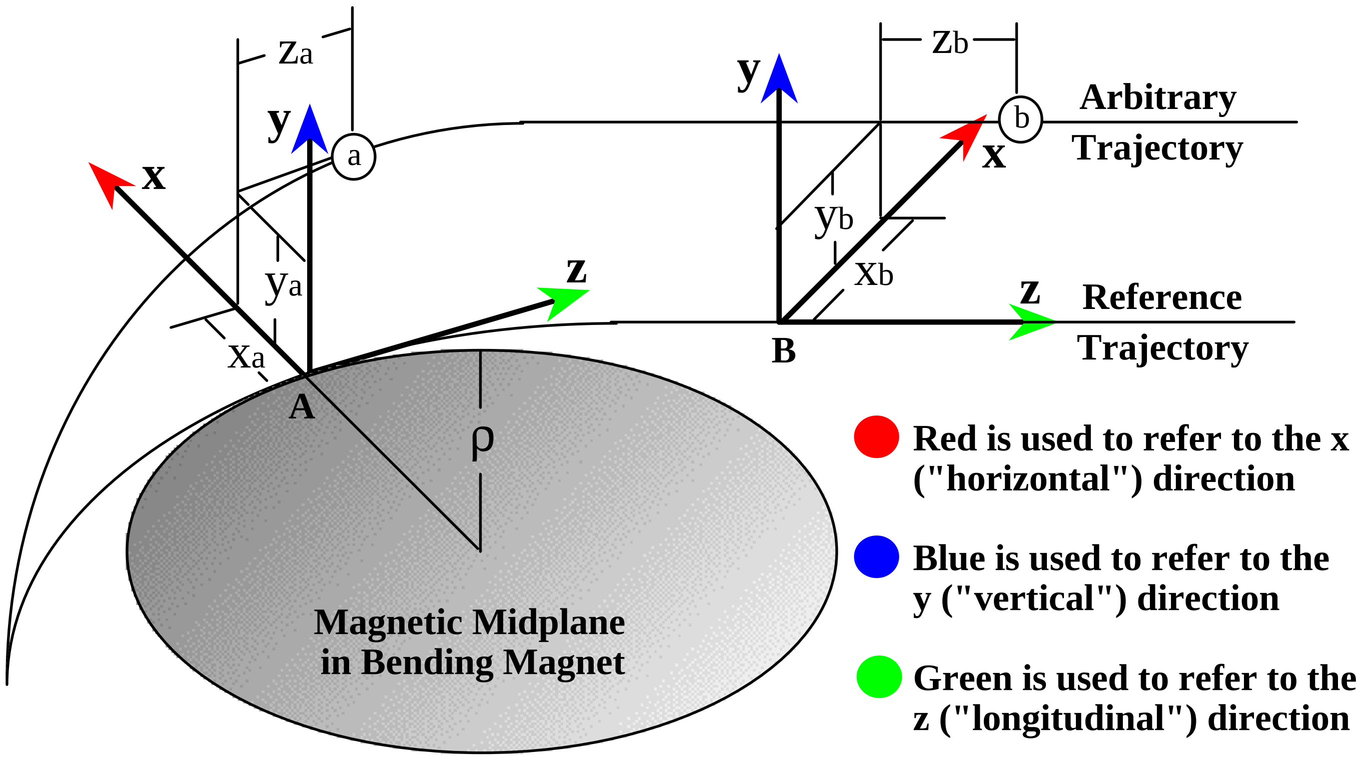
9. Application Modules: TRACE 3-D, TRANSPORT & TURTLE

10. Using TRACE 3-D, TRANSPORT & TURTLE to Solve Some Problems

⇒ **Complete some typical computations with the codes in the class**

1. Basic Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …

- **Particle optics utilizes a perturbation approach to beam dynamics**
- **Motion measured with respect to Reference (or Synchronous) Trajectory**

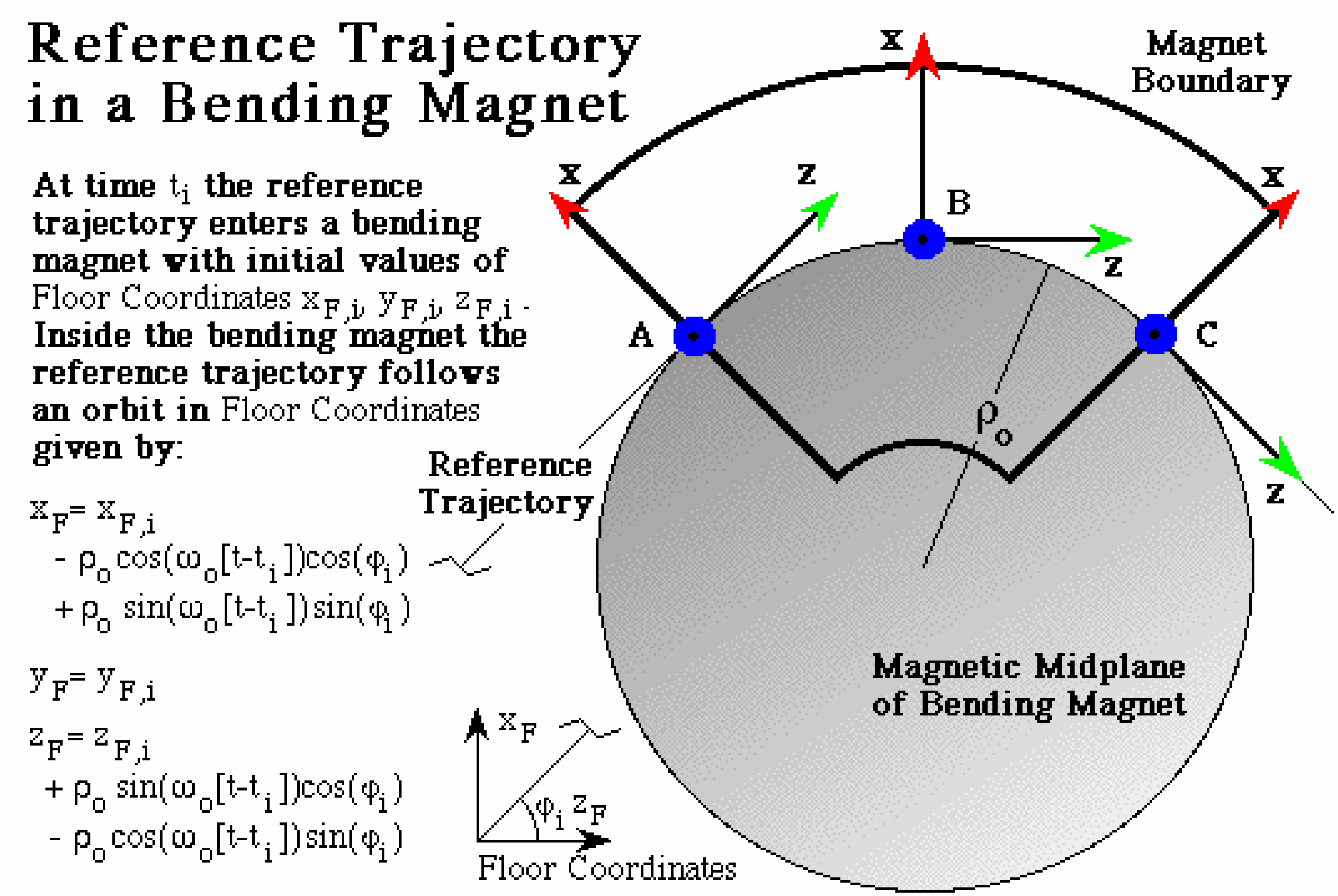


Describing Trajectories and Coordinate Systems

- **Origin of the coordinate system moves with the Reference Trajectory**

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

Reference Trajectory can be thought of as a machine property, often specified in terms of "floor coordinates" (denoted below by subscripts F)



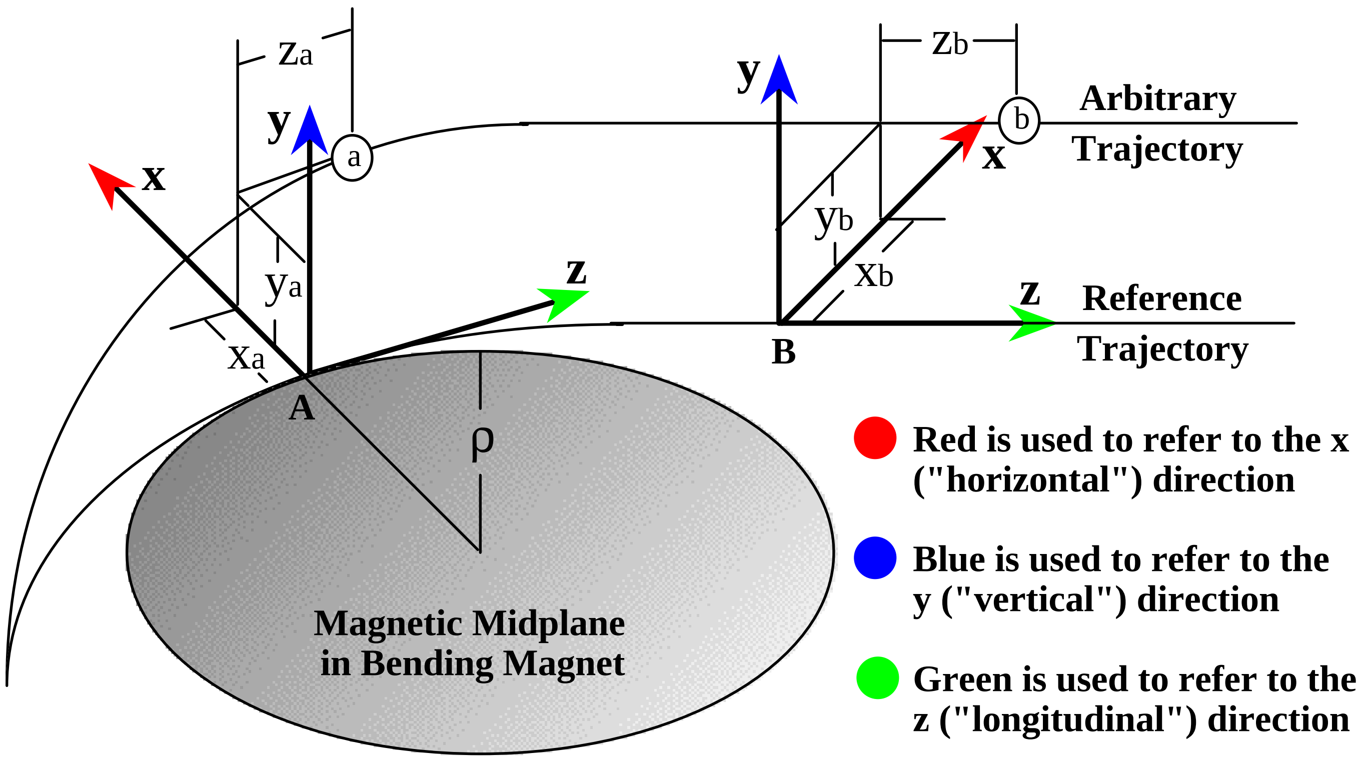
1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

- **In addition to the "floor coordinates" the Reference Trajectory also has a "Reference Velocity" associated with it.**
- **A "Reference Particle" (not necessarily any actual particle) moves along the Reference Trajectory at the Reference Velocity.**
- **Magnitude of the Reference (or Synchronous) Velocity is often denoted $v_s = c\beta_s$, where c = speed of light and β_s is the relativistic speed.**
- **Relativistic parameters can be used to define a Reference $\gamma_s = (1-\beta_s^2)^{-1/2}$, Reference Total Energy $\gamma_s mc^2$, Reference Kinetic Energy $(\gamma_s-1)mc^2$, etc.**
- **Some beam optics codes compute the floor coordinates of the Reference Trajectory but many do not.**

TRANSPORT does compute floor coordinates
TRACE 3-D does not compute floor coordinates

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

So, motion measured with respect to Reference (or Synchronous) Trajectory



Describing Trajectories and Coordinate Systems

⇒ **Particle Optics Codes:** **compute** $[x_b, y_b, z_b]$ **from** $[x_a, y_a, z_a]$

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

- **A map, *M*, can be used to compute [x_b, y_b, z_b] from [x_a, y_a, z_a]. The momentum associated with each coordinate will also be needed, e.g. [Px_a, Py_a, Pz_a].**

- **If we denote the 6-vector [x_b, Px_a, y_b, Py_a, z_b, Pz_a] by [q_i] with i = 1,…,6 then *M* maps [$q_{i\,a}$] into [$q_{i\,b}$]:**

$$[q_{i\,b}] = M\,[q_{i\,a}]$$

- **Since all elements of the 6-vectors [$q_{i\,a}$] and [$q_{i\,b}$] are presumed "small" we should be able to represent the map *M* by a Taylor series expansion:**

$$M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja} + \Sigma_{j\le k}\, \Sigma_k\, T_{ijk}\, q_{ja}q_{ka} + \Sigma_{j\le k\le l}\, \Sigma_{k\le l}\, \Sigma_l\, U_{ijkl}\, q_{ja}q_{ka}q_{la} + \ldots$$

- **In first-order optics, only the first term is used:**

$$[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja}$$

First-order optics ⇒ linear optics

- **In second-order optics, the first 2 terms are used:**

$$[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja} + \Sigma_{j\le k}\, \Sigma_k\, T_{ijk}\, q_{ja}q_{ka}$$

Second- (and higher-)order ⇒ nonlinear optics

- **Less than or equal summations (e.g. $j\le k$) avoid double counting (e.g. $T_{ijk}=T_{ikj}$).**

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

What does this matrix formalism "mean"?

- **Consider particle starting on-axis: $x_a = 0$ and $y_a = 0$ at Reference Velocity [v_s]**
- **The change in the x-coordinate at the end of a system [x_b] due small initial velocities [v_{xa}, v_{ya}] away from the axis can be written as:**

$$x_b = R_{12}\,[x'_a] + R_{14}\,[y'_a]$$

$$\text{with } x'_a = v_{xa} / v_s \approx Px_a / P_s \text{ and } y'_a = v_{ya} / v_s \approx Py_a / P_s$$

- **Suppose we want a lens system that will bring a group of such on axis particles back to the (x) axis. This could be accomplished for <u>ALL</u> v_{xa} & v_{ya} if**

$$R_{12} = R_{14} = 0$$

⇒ **"Point-to-Point" Focus in x** **($R_{12} = 0$)**

Similarly $$R_{32} = R_{34} = 0$$

⇒ **"Point-to-Point" Focus in y** **($R_{34} = 0$)**

Will Return to this Later!

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

- **Useful to break (6-by-6)** R-Matrix **into a set of 9 (2-by-2) submatrices:**

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{bmatrix} = R \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} [R_{xx}] & [R_{xy}] & [R_{xz}] \\ [R_{yx}] & [R_{yy}] & [R_{yz}] \\ [R_{zx}] & [R_{zy}] & [R_{zz}] \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix}$$

- **For a many cases (drifts, quads, solenoids) only three are non-zero:**

$$R_{xx} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix}$$

- **This leads to a "block diagonal"** R-Matrix**:**

$$R = \begin{bmatrix} R_{xx} & 0 & 0 \\ 0 & R_{yy} & 0 \\ 0 & 0 & R_{zz} \end{bmatrix}$$

- **Bending magnets represent an exception to "block diagonal"** R-Matrix
  Bends introduce dispersion: coupling between bend and z,z' planes

1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

Some other matrix properties

- **R_{11} describes the dependence of the output x_b on the input x_a :**

  $R_{11} = M_x$ = x-Magnification ($|R_{11}|>1$) or Demagnification ($|R_{11}|<1$)

  Similarly:

  $R_{33} = M_y$ = y-Magnification ($|R_{33}|>1$) or Demagnification ($|R_{33}|<1$)

- **R_{21} describes the dependence of the output angle x'_b on the input x_a :**

  $R_{21} = - 1 / f_x$ where f_x = x-Focal Length

  Similarly:

  $R_{43} = - 1 / f_y$ where f_y = y-Focal Length

  If $R_{21} < 0$ then focusing in x direction, while $R_{21} > 0$ is defocusing in x direction
  If $R_{43} < 0$ then focusing in y direction, while $R_{43} > 0$ is defocusing in y direction

- **For a Quadrupole, x and y not same ($R_{21} \neq R_{43}$) $\Rightarrow$ astigmatic lens**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Representations

Beam is a Collection ("Ensemble") of Particles

⇒ **Can Certainly Apply Single Particle Equations of Motion to All Particles**

- **Some Particle Optics Codes do this (e.g. TURTLE)**
- **For Design & Other Studies Really Want:**

  Computation Methods for Beam Properties

  Faster Computation

We want a method for Describing a Beam and procedures for simulating the evolution of that description

⇒ **Phase Space Descriptions of Beam Distributions**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **TRANSPORT is a 3rd Order "Matrix" Code - What Does It Calculate?**

  Does It Calculate $[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j R_{ij}\, q_{ja} + \Sigma_{j\leq k}\,\Sigma_k\, T_{ijk}\, q_{ja} q_{ka} + \ldots$ **?**

  No ⇒ TRANSPORT Does Not Advance Individual Particles

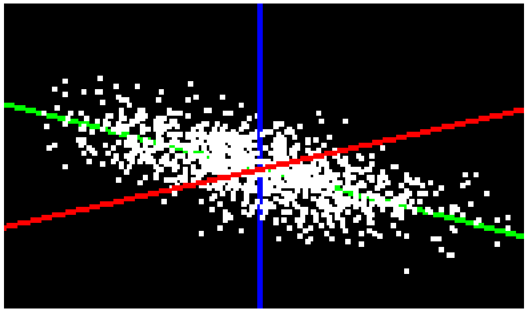
- **TRANSPORT Advances the Beam Distribution's 1st & 2nd Moments**
  - **Beam Described by 1st & 2nd Moments of the Particle Distribution**
  - **1st Moments of the Particle Distribution are Beam Centroids**
  - **2nd Moments of the Particle Distribution are a Matrix (σ Matrix)**

- **Let a Beam Be Described by a Distribution Function** f**:**

$$f = f(x,x',y,y',z,z')$$

with normalization:

$$\int f(x,x',y,y',z,z')\, dx dx' dy dy' dz dz' = 1$$



- **The Distribution Function** f **gives the Particle Density in Phase Space**
- **The Longitudinal Variables (**z,z'**) Are Understood to Mean (**l,δ**)**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **First Moment for** $\langle x \rangle$ **of the Distribution Function** f**:**

$$\langle x \rangle = \int x \, f(x,x',y,y',z,z') \, dx dx' dy dy' dz dz'$$

- **Similar Definitions for** $\langle x' \rangle$, $\langle y \rangle$, $\langle y' \rangle$, $\langle l \rangle$, $\langle \delta \rangle$

- **The Beam Centroid Vector** $[\mathbf{q_i}]_c$ **is Given by 1st Moments:**

$$[\mathbf{q_i}]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = (\langle x \rangle, \langle x' \rangle, \langle y \rangle, \langle y' \rangle, \langle l \rangle, \langle \delta \rangle)$$

- **If the Beam Centroid Follows the Reference Trajectory Then**

$$[\mathbf{q_i}]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = 0$$

- **Reference Trajectory = Optical Component "Central" Axis**
  $\Rightarrow$ **Fields are Expanded About that Central Axis**

- **Beam Centroid = Beam Location with Respect to that Central Axis**
  $\Rightarrow$ **Beam 2nd Moments Computed with Respect to Beam Centroid**

[• **Some Works Use "Centroid" & "Reference" Trajectory Interchangeably**]

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Second Moments Defined by Quadratic Forms of Variables:**

$$\langle x^2 \rangle = \int (x)^2 \, f(x,x',y,y',z,z') \, dx dx' dy dy' dz dz'$$

where we assume that centroid has been removed ($\langle x^2 \rangle \equiv \langle (x - x_c)^2 \rangle$**)**

- **Again, Similar Definitions for** $\langle xx' \rangle$**,** $\langle xy \rangle$**,** $\langle xy' \rangle$**,** $\langle xl \rangle$**,** $\langle x\delta \rangle$**, …**

- **Second Moments Can Be Written as a 6-by-6 Matrix, the σ Matrix:**

$$\sigma_{ij} = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle & \langle xz \rangle & \langle xz' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle & \langle x'y \rangle & \langle x'y' \rangle & \langle x'z \rangle & \langle x'z' \rangle \\ \langle yx \rangle & \langle yx' \rangle & \langle y^2 \rangle & \langle yy' \rangle & \langle yz \rangle & \langle yz' \rangle \\ \langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'^2 \rangle & \langle y'z \rangle & \langle y'z' \rangle \\ \langle zx \rangle & \langle zx' \rangle & \langle zy \rangle & \langle zy' \rangle & \langle z^2 \rangle & \langle zz' \rangle \\ \langle z'x \rangle & \langle z'x' \rangle & \langle z'y \rangle & \langle z'y' \rangle & \langle z'z \rangle & \langle z'^2 \rangle \end{bmatrix}$$

- **The σ Matrix, aka "Beam Matrix", is Symmetric (e.g.** $\langle xx' \rangle = \langle x'x \rangle$**)**

- **If Particle Coordinates Transform as** $[q_{i\,b}] = \Sigma_j R_{ij} q_{ja} \equiv R[q_{i\,a}]$
  It Can Be Shown that the Sigma Matrix $[\sigma_{ij\,b}]$ **Transforms as:**

$$[\sigma_{ij\,b}] = \Sigma_k R_{ik} \Sigma_m R_{mj} [\sigma_{km\,a}] \equiv R[\sigma_{ij\,a}]R^T$$

where R^T **is the Transpose of R.**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **The σ Matrix is Symmetric $\Rightarrow$ Use a "Reduced" Representation**
- **Define Correlation Parameters by:**

$$r_{12} \equiv r_{xx'} = \langle xx' \rangle / (\langle xx \rangle \langle x'x' \rangle)^{1/2} = \sigma_{xx'} / (\sigma_{xx}\, \sigma_{x'x'})^{1/2}$$

$$r_{13} \equiv r_{xy} = \langle xy \rangle / (\langle xx \rangle\, \langle yy' \rangle)^{1/2} = \sigma_{xy} / (\sigma_{yy}\, \sigma_{xy})^{1/2}$$

etc. for the complete set of r_{ij} for $1 \leq i,j \leq 6$

- **From σ-matrix properties:** $r_{ij} \equiv r_{ji}$ **and** $r_{ii} \equiv 1$ **all <u>unitless</u>**
- **TRANSPORT Writes the "Reduced" σ-Matrix as a Lower Half-Matrix:**

$$\sigma_{reduced} = \begin{bmatrix} \langle x^2 \rangle^{1/2} & & & & & \\ \langle x'^2 \rangle^{1/2} & r_{21} & & & & \\ \langle y^2 \rangle^{1/2} & r_{31} & r_{32} & & & \\ \langle y'^2 \rangle^{1/2} & r_{41} & r_{42} & r_{43} & & \\ \langle z^2 \rangle^{1/2} & r_{51} & r_{52} & r_{53} & r_{54} & \\ \langle z'^2 \rangle^{1/2} & r_{61} & r_{62} & r_{63} & r_{64} & r_{65} \end{bmatrix}$$

- **The First Column Contains the RMS Beam <u>Envelopes</u> ("Sizes")**
  - Text Output Writes UNITS Immediately After 1st Column
- **There Are 15 Correlation Coefficients (**r_{ij} with $i<j$, and j running from 1 to 5**)**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Similar to the R-Matrix, the Sigma-Matrix is Often "Block Diagonal"**
- **Can Then Write the σ-Matrix as the Three (non-zero) Submatrices:**

$$\sigma = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

- **Where Each Submatrix is a 2×2 Matrix:**

$$\sigma_{xx} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \qquad \sigma_{yy} = \begin{bmatrix} \sigma_{33} & \sigma_{34} \\ \sigma_{43} & \sigma_{44} \end{bmatrix} \qquad \sigma_{zz} = \begin{bmatrix} \sigma_{55} & \sigma_{56} \\ \sigma_{65} & \sigma_{66} \end{bmatrix}$$

- **Symmetry of σ-Matrix (e.g. $\sigma_{21} = \sigma_{12}$) Means 3 Independent Parameters for Each 2×2 Matrix. So, if it Proves Useful, Can Write Each in Form such as:**

$$\sigma_{xx} = \varepsilon_x \begin{bmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{bmatrix} \qquad \textbf{with } \beta_x\gamma_x - \alpha_x^2 = 1$$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Motion of a particle moving under linear restoring force (harmonic oscillator) can be described in terms of amplitude and phase variables by:**

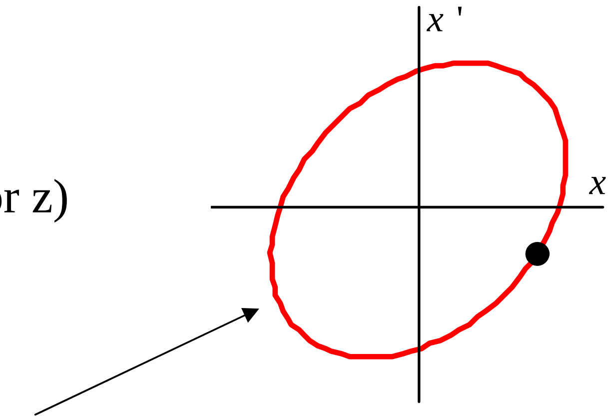
$$x(s) = [\beta_x \varepsilon_x]^{1/2} \cos\big(\psi_x(s) + \psi_x(0)\big)$$

where β_x **is constant and** $\psi_x(s)$ **is linear in** s**:**

$\beta_x = x(0)^2 / \varepsilon_x \qquad \psi_x(s) = k_x s \qquad$ (think of s as t or z)

- β_x **is the amplitude and** $\psi_x(s)$ **is the phase**

- **As** s **increases the particle traces an ellipse in Phase Space**



Phase Space

- **If the force "constant"** k_x **is not a constant, but changes with** s**, e.g.** $k_x(s)$**, the motion can still be described in terms of amplitude and phase:**

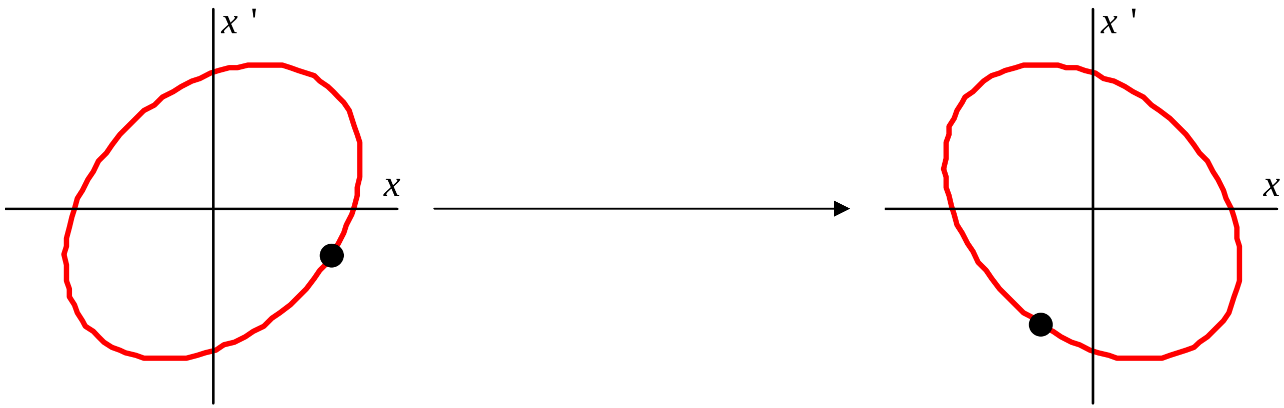
$$x(s) = [\beta_x(s) \varepsilon_x]^{1/2} \cos\big(\psi_x(s) + \psi_x(0)\big)$$

- **Now** $\beta_x(s)$ **is not constant and is** $\psi_x(s)$ **nonlinear:**

$$d\beta_x(s)/ds = -2\alpha_x(s) \qquad \psi_x(s) = \int [\beta_x(s)]^{-1}\, ds$$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

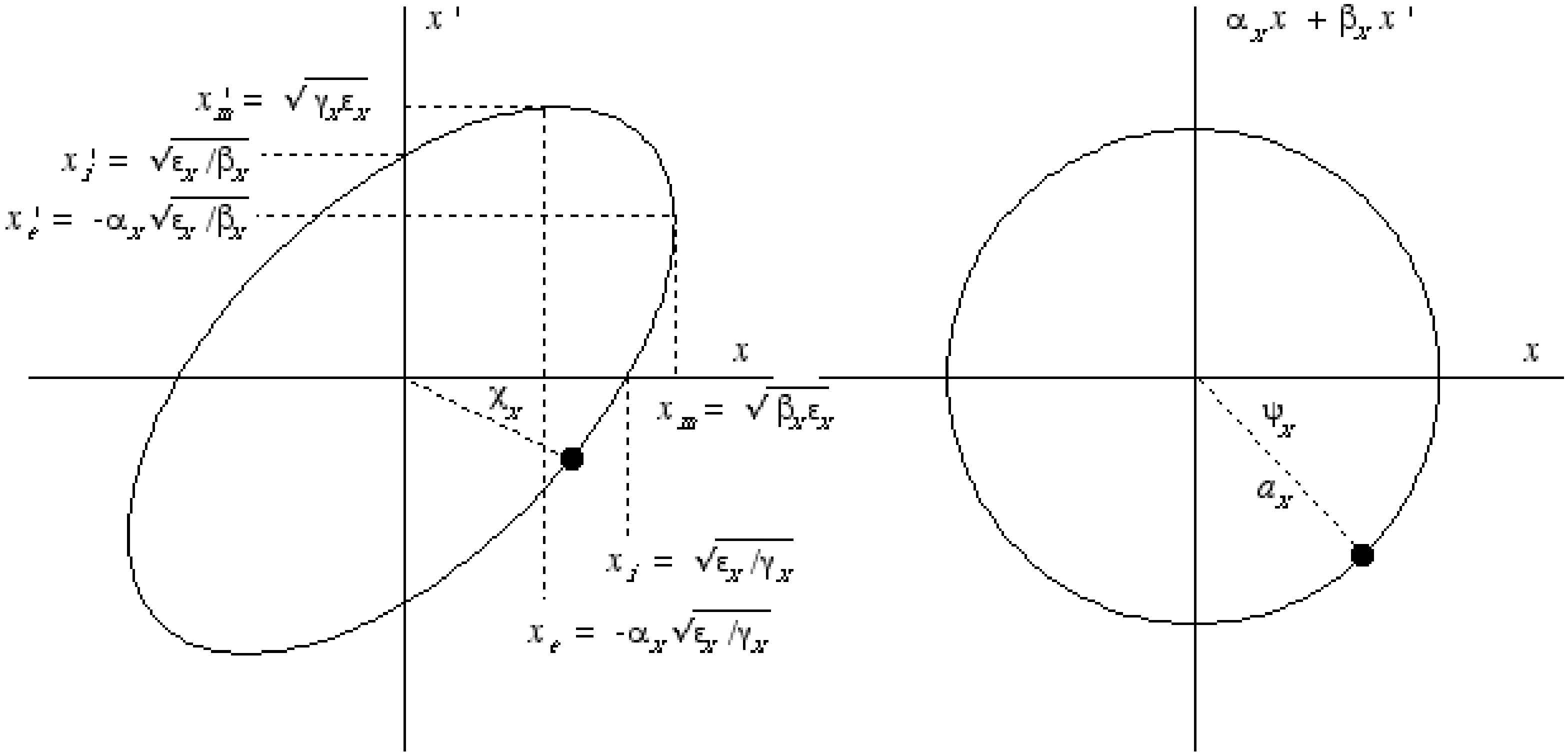
- **The function $\beta_x(s)$ is referred to as the amplitude function**
- **The function $\psi_x(s)$ is referred to as the phase advance**
- **As s increases the particle will still remain on the ellipse, but now the ellipse will change in phase space**



- **If you start with an collection of particles ("beam") with a set of initial amplitudes and phases inside of a given ellipse the ellipse will evolve ("down the beamline") and all particles will remain within that ellipse.**

⇒ Representation Provides a Useful Way to Describe a Beam
(at least for linear optics)

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

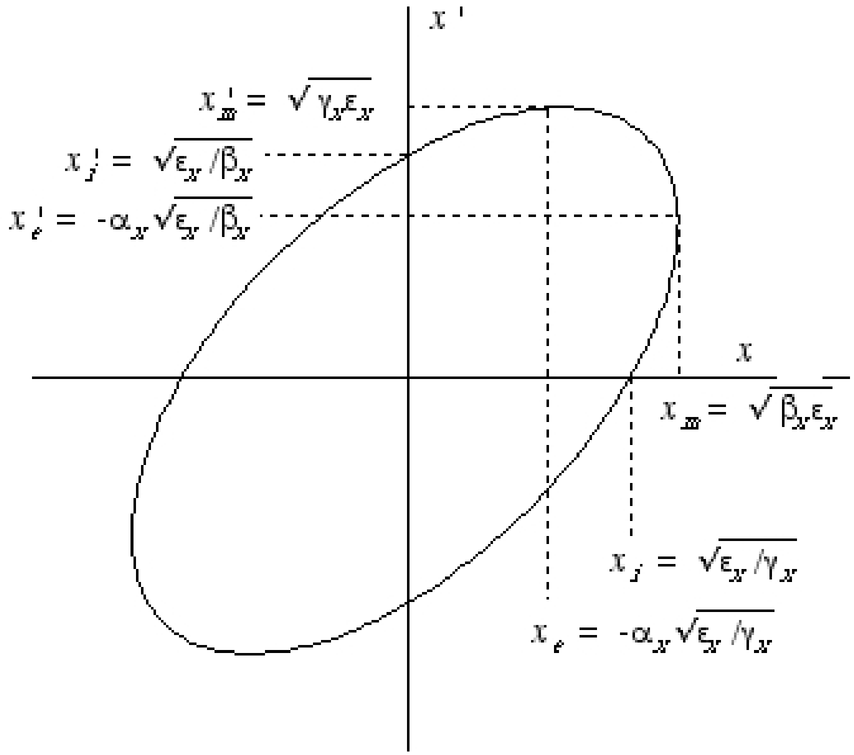


$$\beta_x \varepsilon_x = (a_x)^2$$

$$\tan\chi_x = (\tan\psi_x - \alpha_x) / \beta_x$$

alternatively: $\cos\psi_x = \cos\chi_x / [(\cos\chi_x)^2 + (\alpha_x \cos\chi_x - \beta_x \sin\chi_x)^2]^{1/2}$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)



$$\beta_x \varepsilon_x \equiv \sigma_{11} = \langle x^2 \rangle = (x_{max})^2$$

$$\gamma_x \varepsilon_x \equiv \sigma_{22} = \langle x'^2 \rangle = (x'_{max})^2$$

$$\alpha_x \varepsilon_x \equiv \sigma_{12} = \langle xx' \rangle = r_{12} \, [\sigma_{11}\sigma_{22}]^{1/2} \qquad \text{or } \alpha_x = -1 / [1- r_{12}^2]^{1/2}$$

$$r_{12} \equiv \sigma_{12}/[\sigma_{11}\sigma_{22}]^{1/2} = r_{21} \equiv \sigma_{21}/[\sigma_{11}\sigma_{22}]^{1/2}$$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

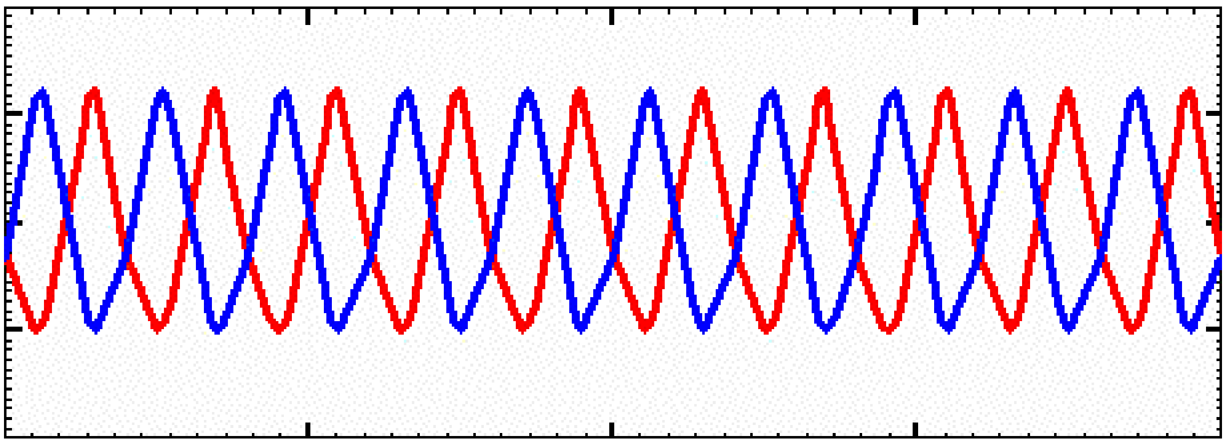
- **The parameters α_x, β_x and γ_x are often called "Twiss Parameters"**

  $\Rightarrow$ **I (try to) use the term Twiss Parameters when describing beam properties**

- **The α_x, β_x and γ_x are also called the "Courant-Snyder Parameters"**

  $\Rightarrow$ **I use Courant-Snyder Parameters when describing machine properties**

- **For a periodic (stable) system, the functions $\alpha_x(s)$ and $\beta_x(s)$ will be periodic**



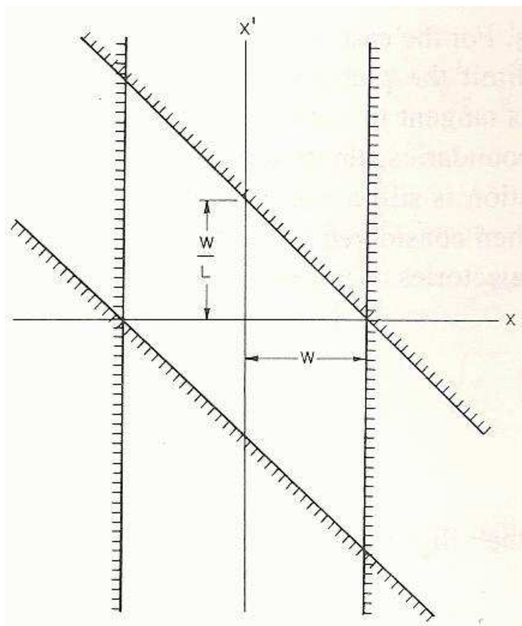
Example of $\beta_x(s)$ and $\beta_y(s)$ for a storage ring

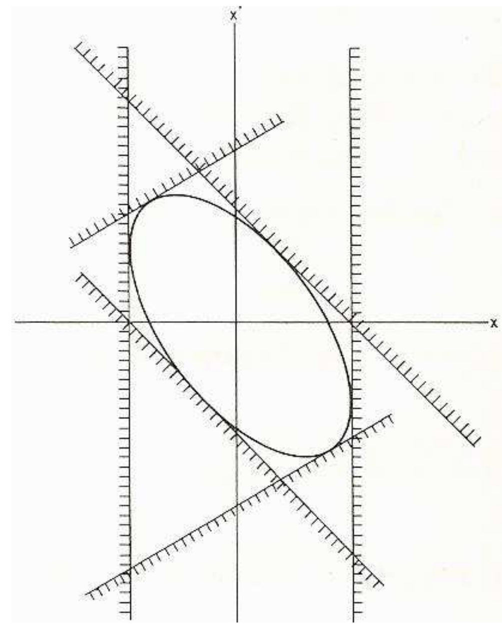
- **The only beam that will survive many passes through such a system (i.e. many turns around the ring) is called the "matched beam"**

  $\Rightarrow$ **The "matched beam" requirements are machine properties**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

- **Two slits will define an "acceptance" phase space for a beam**
- **Slit-defined "acceptance" phase space also specifies an enclosing ellipse**





Phase space accepted by two slits, each of width W, separated by L.

Phase space accepted by three slits, with focusing elements in between.

[Figures from page 101 of *The Optics of Charged Particle Beams* by David C. Carey, Harwood Academic Publishers (1987).]

- **An inscribed "acceptance" ellipse can be used to describe accepted beam**

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

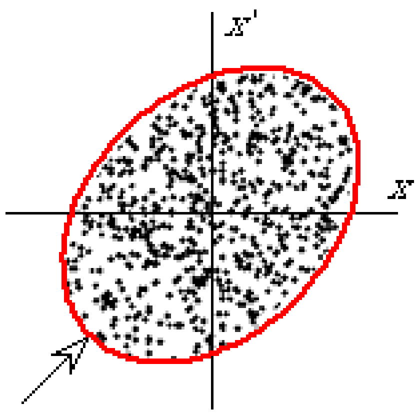
Emittances: RMS, Boundary, Normalized, Unnormalized, …

⇒ **The value of ε_x, which is a measure of the ellipse area, is the x-emittance.**

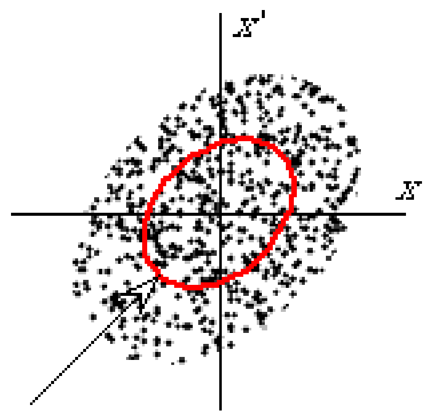
One convention defines ε_x as the area of the ellipse divided by π, and written as

$$\varepsilon_x = [\langle x^2\rangle \langle x'^2\rangle - \langle x'x \rangle] = 0.1\ \pi\text{-mm-mrad} \qquad (\pi \text{ in the units})$$

- **Emittance Definitions & Usage are Not Uniform**
- **Laboratory Emittance = "unnormalized" emittance (assumption so far)**
- **Emittance ellipse may describe all, or only a part, of the beam**



Enclosing Ellipse:
"boundary" emittance



Partially Enclosing Ellipse:
"RMS" emittance, "50% contour" emittance, ...

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

Emittances: RMS, Boundary, Normalized, Unnormalized, …

$$\varepsilon_x = [\ \langle x^2 \rangle \langle x'^2 \rangle - \langle x'x\ \rangle] = 0.1\ \pi\text{-mm-mrad}$$ (π **in the units**)

- **Laboratory Emittance (i.e. "unnormalized") not preserved with acceleration**

$$\varepsilon_x(E_b) = \varepsilon_x(E_a)\ [\beta_a\gamma_a] \,/\, [\beta_b\gamma_b]$$

- **Normalize emittance, $\varepsilon_{x,n}$, is preserved with acceleration (and linear optics)**

$$\varepsilon_{x,n} = [\beta_a\gamma_a]\ \varepsilon_x(E_a)$$

- **Laboratory Emittance scales with kinetic energy** $E_a \rightarrow E_b$

$$\varepsilon_x(E_b) = \varepsilon_x(E_a)\ [\beta_a\gamma_a] \,/\, [\beta_b\gamma_b] \approx \varepsilon_x(E_a)\ [E_a \,/\, E_b]^{1/2}$$

- **Reported emittance may also need to be scaled with mass** $m_a \rightarrow m_b$

$$\varepsilon_x(m_b) = \varepsilon_x(m_a)\ [\beta_a\gamma_a] \,/\, [\beta_b\gamma_b] \approx \varepsilon_x(m_a)\ [m_a \,/\, m_b]^{1/2}$$

2. Describing a Beam - Phase Space, Semi-Axes & Twiss Rep. …(cont'd)

Emittances: RMS, Boundary, Normalized, Unnormalized, …

- **Example:** HVEE Model SO-100-1 ion source:
- **100 μA Boron (B^+) beam, emittance at 30 keV extraction is quoted as:**

$$\varepsilon = 15 \text{ mm-mrad (MeV)}^{1/2}$$

- **Explicit energy dependence indicates this is a normalized emittance**
- **No factor of π included in units explicitly, but most ion source specs give the emittance as the product of the ellipse semi-axes. Thus expect:**

$$\varepsilon_{lab} = 1.5\ \pi\text{-mm-mrad } (1 \text{ MeV}/E_{lab}[\text{MeV}])^{1/2}$$

- **For 10 keV extraction, the emittance to use in calculations is thus:**

$$\varepsilon_{lab}(10 \text{ keV } B^+) = 1.5\ \pi\text{-mm-mrad } (1.0 / 0.01)^{1/2+}) = 15.0\ \pi\text{-mm-mrad}$$

- **For 10 keV Phosphorous (P^+) apparently need to scale with mass**

$$\varepsilon_{lab}(10 \text{ keV } P^+) = 15.0\ \pi\text{-mm-mrad } (m[B]/m[P])^{1/2} = [0.3490]^{1/2} 15.0\ \pi\text{-mm-mrad}$$
$$= 8.86\ \pi\text{-mm-mrad}$$

⇒ **Brightness ∝ I/ε^2 and has at least as many variants as emittance**

3. Equations of Motion: Drifts, Quads, Bends

But how do we get from $\underline{F} = m\underline{a}$ to the matrix formalism?

$\Rightarrow$ **Equations of Motion - Simple Version**

3. Equations of Motion: Drifts, Quads, Bends

- **Classical mechanics, Newton's 2nd Law:**

  $\mathbf{F} = d\mathbf{p}/dt$ **(F, p 3-vectors)**

- **Relativistically correct, with proper interpretation of F and p (need a 4-vector)**

\- **Spatial components:**

$$F_x = dp_x/dt \quad \text{with } p_x = \beta_x \gamma mc$$
$$F_y = dp_y/dt \quad \text{with } p_y = \beta_y \gamma mc$$
$$F_z = dp_z/dt \quad \text{with } p_z = \beta_z \gamma mc$$

\- **4th component (energy W):**

$$\mathbf{F} \cdot \mathbf{v} = dW/dt \quad \text{with } W^2 = p^2c^2 + m^2c^4$$

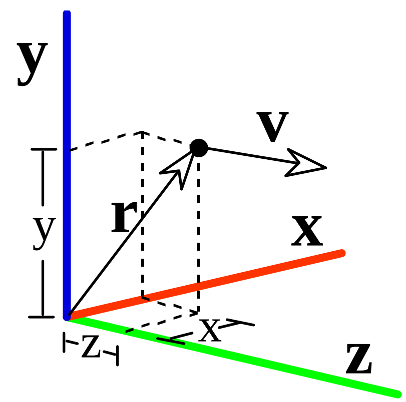
- **More elegant formulation uses Hamiltonian mechanics (not discussed further here)**

Equations of Motion - Simple Version

This section of the lecture uses:

BoldFont for 3-Vectors

Plain Font for Scalars



The z **coordinate is often denoted by** *l*

l **= path length difference**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Simple Version (con't)

- **For cases where the Reference Trajectory is straight, and there is no acceleration (i.e. all magnetic elements except bends), the equations of motion in (TRANSPORT, TRACE 3-D) coordinates can be derived using the relation:**

$$d\,/dt = (ds/dt)\; d\,/ds \equiv c\beta_s\; d\,/ds, \qquad \text{e.g. } v_X \equiv dx/dt = c\beta_s dx/ds \equiv c\beta_s x'$$

- **Transverse motion (x and y):**

$$F_x = dp_x/dt = c\beta_s\; d\,(\beta_x\gamma mc)\,/ds = c\beta_s\; d\,(x'\beta_s\gamma mc)\,/ds = c\beta_s^{\,2}\,\gamma mc\; dx'/ds$$
$$F_y = dp_y/dt = c\beta_s\; d\,(\beta_y\gamma mc)\,/ds = c\beta_s\; d\,(y'\beta_s\gamma mc)\,/ds = c\beta_s^{\,2}\,\gamma mc\; dy'/ds$$
$$\text{where:}\quad x' = dx/ds, \qquad y' = dy/ds$$

- **Convenient to write these in the form:**

$$dx\,/ds = x' \qquad\qquad dx'/ds = [F_x\,/\,p_s]\; 1/(c\beta_s)\;(\gamma_s/\gamma)$$
$$dy\,/ds = y' \qquad\qquad dy'/ds = [F_y\,/\,p_s]\; 1/(c\beta_s)\;(\gamma_s/\gamma)$$

- **Use the Lorentz force to get the forces F_x , F_y for particular fields**

For a force free region (e.g. drift space) $F_x = F_y = 0$, hence $dx'/ds = dy'/ds = 0$

So that $dx\,/ds = x' = \text{constant}_x$, and $dy\,/ds = y' = \text{constant}_y$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Simple Version (con't)

- **Longitudinal motion in TRANSPORT coordinates (l and δ), when the Reference Trajectory is straight and there is no acceleration (i.e. all magnetic elements except bends), is simple but non-trivial**
- **For magnetic fields, where $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$, then $\mathbf{F} \cdot \mathbf{v} = 0$, and $dW/dt = 0$**

$$dW/dt = c\beta_s\, dW/ds = c\beta_s\, d(\gamma mc^2)/ds = mc^3\beta_s\, d\gamma/ds = 0$$

If $d\gamma/ds = 0$ **, then** $d\beta/ds = 0$ **also, and likewise** $d(\beta\gamma)/ds = 0$

- **So with no acceleration, then for the Reference Trajectory variables**

$$d(\beta_s\gamma_s)/ds = 0$$

and since $\delta = \left[(\beta\gamma)/(\beta_s\gamma_s)\right] - 1$ **one then has:**

$$d\delta/ds = 0 \qquad (\Rightarrow \text{Conservation of Energy, Bends too})$$

- **More general case (e.g. with acceleration) one can show that**

$$\frac{d\delta}{dz_s} = \frac{1}{(1+\delta)}\frac{\beta_z\gamma}{\beta_s\gamma_s}\frac{1}{c\beta_s p_s}\left[F_z + \left(\frac{v_x}{v_z}\right)F_x + \left(\frac{v_y}{v_z}\right)F_y\right] - \frac{(1+\delta)}{\beta_s\gamma_s}\frac{d(\beta_s\gamma_s)}{dz_s}$$

- **What about the longitudinal coordinate l ?**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Simple Version (con't)

- l **is the projected (on z direction) path length difference**

$$dl/dt = c\beta_s\, dl/ds \;,\quad \textbf{but} \quad dl/dt = v_z - v_s = c(\beta_z - \beta_s), \quad \textbf{hence}$$

$$dl/ds = (\beta_z/\beta_s) - 1$$

- **Longitudinal velocity** β_z **(<u>not</u> <u>conserved</u>) in terms of other variables (**x', y', δ**)**

$$\beta_z = \left\{(\beta)^2 - (\beta_x)^2 - (\beta_y)^2\right\}^{1/2} = \left\{(\beta)^2 - (x'\beta_s)^2 - (y'\beta_s)^2\right\}^{1/2}$$

it can be shown that $\beta = \beta_s\,(1+\delta) / \left[1 + \delta(2+\delta)(\beta_s)^2\right]^{1/2}$ **, hence**

$$\beta_z = \beta_s \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - (x')^2 - (y')^2\right\}^{1/2} \;, \textbf{ so finally:}$$

$$dl/ds = \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1$$

- **Note that** $dl'/ds \neq 0$ **, where** $l' = dl/ds$**, but has an "apparent" "force"** F_l **:**

$$F_l = (d/ds)\left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2}$$

- **Longitudinal coordinate** l **important for radiofrequency (RF) components**

[• **Not all codes use the (TRANSPORT, TRACE 3-D) longitudinal coordinate** l]

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Drift

For (non dipole) magnetic systems without acceleration, the equations of motion in TRANSPORT, TRACE 3-D variables x, x', y, y', l, δ **are:**

$$dx/ds = x' \qquad dx'/ds = [F_x / p_s]\, 1/(c\beta_s) \left(1/\left\{1 + \delta(2+\delta)(\beta_s)^2\right\}^{1/2}\right)$$

$$dy/ds = y' \qquad dy'/ds = [F_y / p_s]\, 1/(c\beta_s) \left(1/\left\{1 + \delta(2+\delta)(\beta_s)^2\right\}^{1/2}\right)$$

$$dl/ds = \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1 \qquad d\delta/ds = 0$$

Return to the simplest example: Drift (field free region):

$$dx/ds = x' \qquad dx'/ds = 0$$

$$dy/ds = y' \qquad dy'/ds = 0$$

$$dl/ds = \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1 \qquad d\delta/ds = 0$$

Integrate ds **from** s_a **to** s_b**, with** $s_b - s_a = L$**, the length of the drift, the solutions are:**

$$x_b = x_a + x'_a L \qquad x'_b = x'_a$$

$$y_b = y_a + y'_a L \qquad y'_b = y'_a$$

$$l_b = l_a + \left\{\left((1+\delta_o)^2 / \left[1 + \delta_o (2+\delta_o)(\beta_s)^2\right]\right) - x'^2_a - y'^2_a\right\}^{1/2} L - L \qquad \delta_b = \delta_a \equiv \delta_o$$

$\Rightarrow$ **Drift is inherently nonlinear for the longitudinal coordinate**

$$l_b \cong l_a + (\delta_o/\gamma_s^2) L + O(\delta_o^2, x'^2_a, y'^2_a) + \text{higher order terms}$$

[• **Although codes may not use the (TRANSPORT, TRACE 3-D) longitudinal coordinate** l**, their** **<u>equivalent</u>** **longitudinal coordinates are still nonlinear**]

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Drift (con't)

- **Solution for Drift is in the form of matrix equation, taking an initial vector** $[q_{i\,a}] = (x_o, x'_o, y_o, y'_o, l_o, \delta_o)$ **to a final vector** $[q_{i\,b}] = (x, x', y, y', l, \delta)$**, where the** R-Matrix **equation,** $q_{i\,b} = [R]\, q_{i\,a}$**, is obtained using only** $l = (\delta_o/\gamma_s^2)\,L + l_o$.
- **Useful to break (6-by-6)** R-Matrix **into a set of 9 (2-by-2) submatrices:**

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{bmatrix} = R \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} [R_{xx}] & [R_{xy}] & [R_{xz}] \\ [R_{yx}] & [R_{yy}] & [R_{yz}] \\ [R_{zx}] & [R_{zy}] & [R_{zz}] \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix}$$

- **For a drift, most submatrices are zero, only three are non-zero:**

$$R_{xx} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & s/\gamma_s^2 \\ 0 & 1 \end{bmatrix}$$

- **For a drift of length** L**, then the** R-Matrix **above is evaluated for** $s = L$
- **Easy to show that, for two drifts of lengths** L_1 **and** L_2**, the multiplication of the two** R-Matrices **is simply a** R-Matrix **for length** $L = L_1 + L_2$

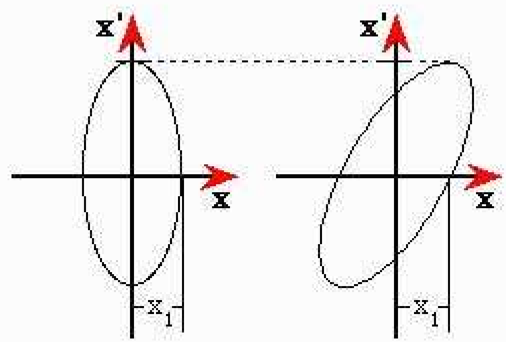
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

R-Matrix Example - Drift

- **250 keV protons (β = 0.023080 and γ = 1.000266)**
- **Drift Length of 2 Meter**

The Drift Piece is an *Approximate First Order* Optics Element

The non-trivial sub matrices for the Drift Piece are determined by the value of the Effective Drift Length L in the Piece Window, and the relativistic energy γ. These submatrices are:



$$[R_{xx}] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 2.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$$

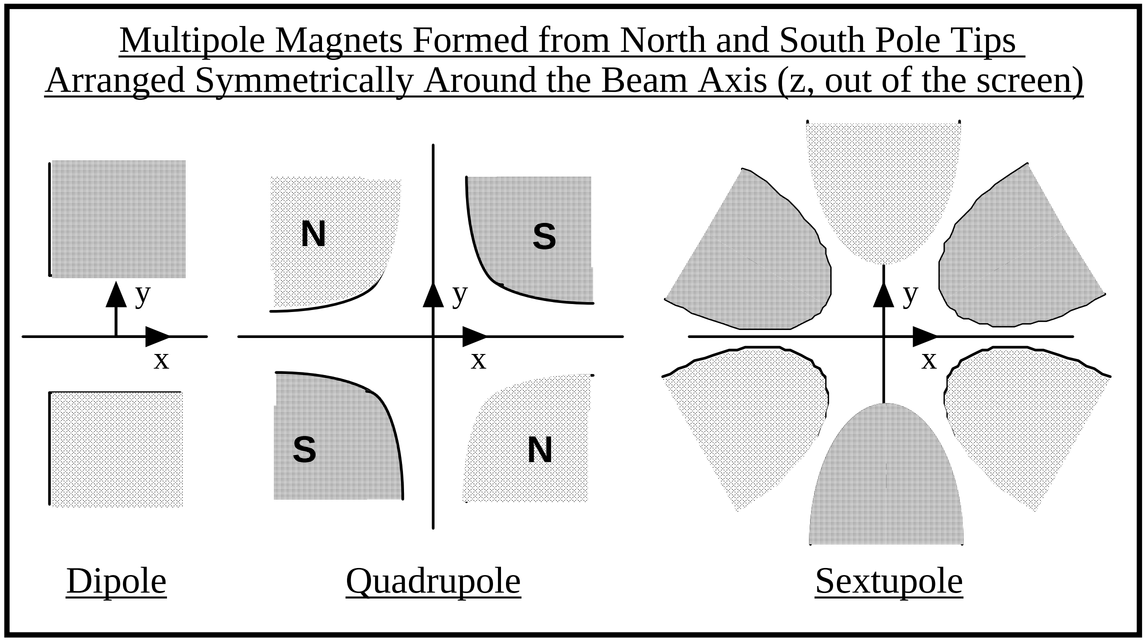
$$[R_{yy}] = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 2.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$$

$$[R_{zz}] = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} = \begin{bmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 1.9989 \\ 0.0000 & 1.0000 \end{bmatrix}$$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole

- **Magnetic quadrupole is one example of magnetic cylindrical multipole field**



- **Magnetic Quadrupole Field: inside:** $\mathbf{B} = (B_o/a)\,[(\mathrm{r}\sin\theta)\,\mathbf{x} + (\mathrm{r}\cos\theta)\,\mathbf{y}]$ $(0<z<L)$
  (in absence of fringe fields) outside: $\mathbf{B} = 0$ $(z<0 \text{ or } z>L)$
- **Lorentz Force Gives** F_x **and** F_y**:** $\mathbf{F} = (q)\,[\mathbf{E} + \mathbf{v}\times\mathbf{B}] = (q)\,\mathbf{v}\times\mathbf{B}$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

Key Parameters

For Optics:

Pole Tip Field, B_o

Bore Radius, $a\ (=r_1)$

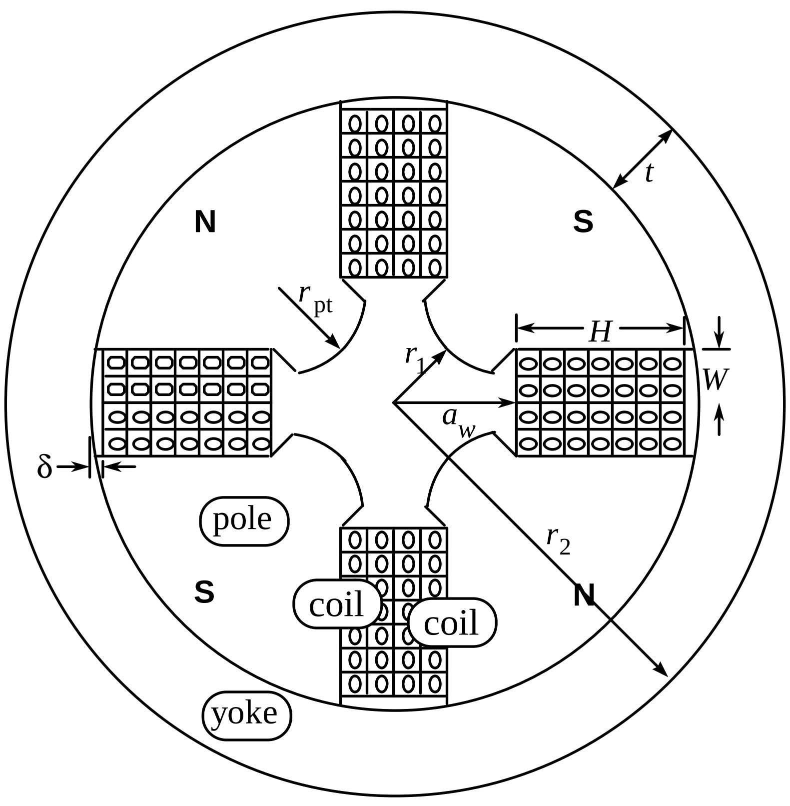
Effective Length, L

For Engineering:

No. of Turns/Coil, n

Current in Coil, I

Typical Design of an Electromagnetic Quadrupole Showing Key Parameters Defining the Pole-Tip Geometry and Field Coils.



3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

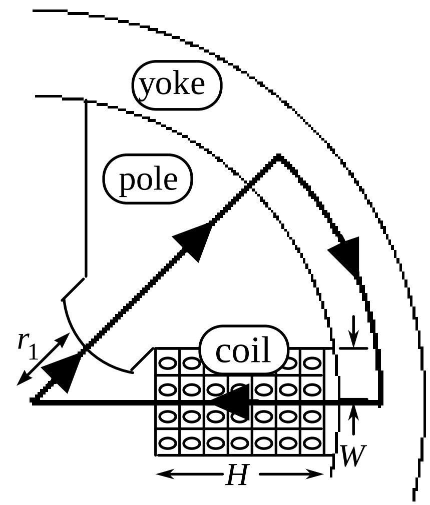
Equations of Motion - Magnetic Quadrupole (con't)

To Relate Engineering & Optics Parameters: Use Ampere's Law

Separate Integral into Four Parts

$$\oint \boldsymbol{H} \bullet \boldsymbol{ds} = I(1) + I(2) + I(3) + I(4)$$

$$\oint \boldsymbol{H} \bullet \boldsymbol{ds} = nI$$



Integration Path

I(1) **Main Contribution from Bore Radial Integral**

$$I(1) = \int_0^{r_1} H_{pt} dr = \int_0^{r_1} (B_o/\mu_o)(r/r_1) dr = B_o r_1/(2\mu_o)$$

I(2)+ I(3) **Contributions of Pole and Yoke Small (μ >>1)**

$$I(2) = \int_{r_1}^{r_{int}} \boldsymbol{H}_{pole} \bullet \boldsymbol{dr} = (1/\mu_{pole}) \int_{r_1}^{r_{int}} \boldsymbol{B}_{pole} \bullet \boldsymbol{dr}$$

$$I(3) = (r_{int}) \int_0^{\pi/4} \boldsymbol{H}_{yoke} \bullet \boldsymbol{d\Theta} = (1/\mu_{yoke})(r_{int}) \int_0^{\pi/4} \boldsymbol{B}_{yoke} \bullet \boldsymbol{d\Theta}$$

I(4) **Contribution Vanishes ($\boldsymbol{H} \perp \boldsymbol{dr}$)**

$$I(4) = \int_{r_{int}}^{0} \boldsymbol{H} \bullet \boldsymbol{dr} = 0 \quad \Rightarrow \quad B_O = 2\mu_O\, nI / r_1$$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

- **Magnetic Field: B** = (B_o/a) [(r sinθ) **x** + (r cosθ) **y**] = (B_o/a) [(y) **x** + (x) **y**]**, or**

  $$B_x = (B_o/a)\, y = B'\, y \qquad B_y = (B_o/a)\, x = B'\, x$$

- **Force Components:** $F_x = -\,(qc)\,\beta_z B_y$ **and** $F_y = (qc)\,\beta_z B_x$ **, or**

  $$F_x = -\,(qcB')\,\beta_z x \qquad F_y = (qcB')\,\beta_z y$$

- **Equations of Motion**

  $dx/ds = x'$ $\quad dx'/ds = -\,[(qc)\,\beta_z B'/p_s]\,1/(c\beta_s)\,(\gamma_s/\gamma)\,x = -\,[qB'/p_s](\beta_z/\beta_s)(\gamma_s/\gamma)\,x$

  $dy/ds = y'$ $\quad dy'/ds = +\,[(qc)\,\beta_z B'/p_s]\,1/(c\beta_s)\,(\gamma_s/\gamma)\,y = +\,[qB'/p_s](\beta_z/\beta_s)(\gamma_s/\gamma)\,y$

  $dl/ds = (\beta_z/\beta_s)-1$ $\quad d\delta/ds = 0$ **(longitudinal same as a drift)**

- **Expand non-linear terms** $(\beta_z/\beta_s) \cong 1 + (\delta_o/\gamma_s^2) + \ldots$ **,** $(\beta_z/\beta_s)(\gamma_s/\gamma) \cong 1 - \delta_o + \ldots$

  $dx/ds = x'$ $\quad dx'/ds = -\,[qB'/p_s]\,x = -\,[K_1]\,x$ $\quad$ K_1 **is Quad Coefficient:**

  $dy/ds = y'$ $\quad dy'/ds = +\,[qB'/p_s]\,y = +\,[K_1]\,x$ $\quad$ $K_1 = [qB'/p_s] = (q/p_s)\,[B_o/a]$

  $dl/ds = \delta_o/\gamma_s^2$ $\quad d\delta/ds = 0$ **(longitudinal same as a drift)**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

- **Transverse motion**

  $dx/ds = x'$ $\quad dx'/ds = dx^2/ds^2 = -[K_1]\,x$ **or** $dx^2/ds^2 + [K_1]\,x = 0$

  $dy/ds = y'$ $\quad dy'/ds = dy^2/ds^2 = +[K_1]\,x$ **or** $dy^2/ds^2 - [K_1]\,y = 0$

- **First order: simple harmonic type motion, solutions depend upon the sign of** K_1**. Useful to define** $k = |K_1|^{1/2}$**, then for** $K_1 > 0$**:**

$$x = x_o\cos(ks) + x'_o\sin(ks)/k \qquad x' = -kx_o\sin(ks) + x'_o\cos(ks)$$

$$y = y_o\cosh(ks) + y'_o\sinh(ks)/k \qquad y' = +ky_o\sinh(ks) + y'_o\cosh(ks)$$

$K_1 > 0$ **is an "x-focusing" quad, and x is typically taken to be horizontal direction**

- **For** $K_1 < 0$**, the trigonometric and hyperbolic functions are exchanged:**

$$x = x_o\cosh(ks) + x'_o\sinh(ks)/k \qquad x' = +kx_o\sinh(ks) + x'_o\cosh(ks)$$

$$y = y_o\cos(ks) + y'_o\sin(ks)/k \qquad y' = -ky_o\sin(ks) + y'_o\cos(ks)$$

- **One phase plane is focusing (trigonometric functions) and one phase plane is defocusing (hyberbolic functions)**

$\Rightarrow$ **Need at least 2 quads for focusing in both transverse directions**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Quadrupole (con't)

- **The Result is a Block Diagonal** R-Matrix**:**

$$R = \begin{bmatrix} R_{xx} & 0 & 0 \\ 0 & R_{yy} & 0 \\ 0 & 0 & R_{zz} \end{bmatrix}$$

- **For an x-focusing (**$K_1 > 0$**) quad, the three non-zero submatrices are:**

$$R_{xx} = \begin{bmatrix} \cos(ks) & \sin(ks)/k \\ -k\sin(ks) & \cos(ks) \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} \cosh(ks) & \sinh(ks)/k \\ k\sinh(ks) & \cosh(ks) \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & s/\gamma_s^2 \\ 0 & 1 \end{bmatrix}$$

- **For a quad of length** L**, then the** R-Matrix **above is evaluated for** $s = L$

- **A useful case is where** $kL << 0$**, but** k^2L **is *not* "small", then** R-Matrix **is:**

$$R_{xx} = \begin{bmatrix} 1 & 0 \\ -k^2L & 1 \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} 1 & 0 \\ +k^2L & 1 \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ **Thin Lens approximation with focal lengths $f_x = 1/(k^2L)$ and $f_y = -1/(k^2L)$**

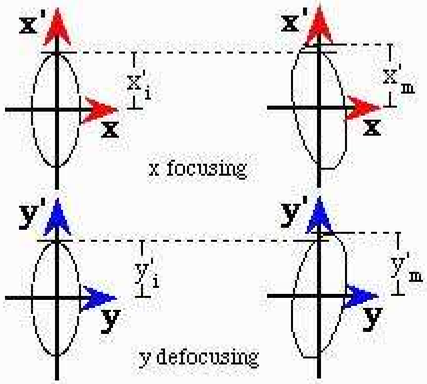
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

R-Matrix Example - Quadrupole 1: QL-100

- **250 keV protons (β = 0.023080 and γ = 1.000266)**
- **Length L of 6 cm, Aperture a of 1.7 cm, Pole Tip Field B_o of 0.034120 T (Gradient B'=20.0706 T/m, Quadrupole Coefficient K_1=277.78 m^{-2}, k=16.667 m^{-1})**

The Quadrupole is an *Approximate First Order* Optics Element

The non-trivial submatrices for the Quadrupole Piece are determined by the value of the Quadrupole Coefficient $K_1 = k^2$, the Effective Length L, and the relativistic energy γ. The submatrices for a x-focusing quadrupole are:



$$[\mathbf{R}_{xx}] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \cos(kL) & \sin(kL)/k \\ -k\sin(kL) & \cos(kL) \end{bmatrix} = \begin{bmatrix} 0.5403 & 0.0505 \\ -14.0246 & 0.5403 \end{bmatrix}$$

$$[\mathbf{R}_{yy}] = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} \cosh(kL) & \sinh(kL)/k \\ k\sinh(kL) & \cosh(kL) \end{bmatrix} = \begin{bmatrix} 1.5431 & 0.0705 \\ 19.5869 & 1.5431 \end{bmatrix}$$

$$[\mathbf{R}_{zz}] = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} = \begin{bmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.0600 \\ 0.0000 & 1.0000 \end{bmatrix}$$

$\Rightarrow$ **Thin Lens approximation gives focal lengths $f_x = -f_y = 0.06$ m**

$\Rightarrow$ **Thick Lens focal lengths $f_x = -1/R_{21} = 0.0713$ m and $f_y = -1/R_{43} = -0.0511$ m**

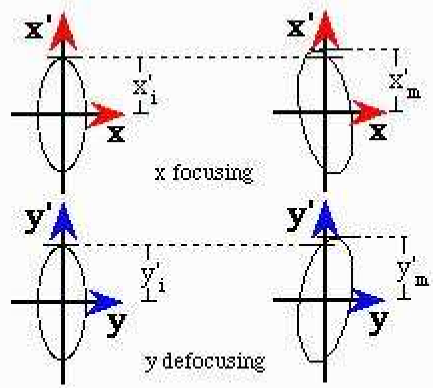
3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

R-Matrix Example - Quadrupole 2: QL-100

- **10 keV Phosphorous (β = 0.000833 and γ = 1.000000)**
- **Length L of 6 cm, Aperture a of 1.7 cm, Pole Tip Field B_o of 0.037840 T (Gradient B' = 22.2588 T/m, Quadrupole Coefficient K_1 = 277.79 m^{-2}, k = 16.667 m^{-1})**

The Quadrupole is an *Approximate First Order* Optics Element

The non-trivial submatrices for the Quadrupole Piece are determined by the value of the Quadrupole Coefficient $K_1 = k^2$, the Effective Length L, and the relativistic energy γ. The submatrices for a x-focusing quadrupole are:



$$[R_{xx}] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \cos(kL) & \sin(kL)/k \\ -k\sin(kL) & \cos(kL) \end{bmatrix} = \begin{bmatrix} 0.5403 & 0.0505 \\ -14.0251 & 0.5403 \end{bmatrix}$$

$$[R_{yy}] = \begin{bmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} \cosh(kL) & \sinh(kL)/k \\ k\sinh(kL) & \cosh(kL) \end{bmatrix} = \begin{bmatrix} 1.5431 & 0.0705 \\ 19.5879 & 1.5431 \end{bmatrix}$$

$$[R_{zz}] = \begin{bmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{bmatrix} = \begin{bmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.0600 \\ 0.0000 & 1.0000 \end{bmatrix}$$

$\Rightarrow$ **Thin Lens approximation gives focal lengths $f_x = -f_y = 0.06$ m**

$\Rightarrow$ **Thick Lens focal lengths $f_x = -1/R_{21} = 0.0713$ m and $f_y = -1/R_{43} = -0.0511$ m**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Some ElectroMagnetic Quadrupole (EMQ) Formulas

- **Pole Tip Magnetic Field**

$$B_o = 2\mu_o\, nI\, / r_1$$

$$B_o(\mathrm{T}) = 8\pi \times 10^{-7}\, [nI(\mathrm{Amps})]/[r_1(\mathrm{m})]$$

- **Quadrupole Gradient**

$$B\,' = B_o / r_1 = 2\mu_o\, nI\, / (r_1)^2$$

$$B\,'(\mathrm{T/m}) = 8\pi \times 10^{-7}\, [nI(\mathrm{Amps})]/[r_1(\mathrm{m})]^2$$

- **Quadrupole Strength**

$$\kappa \equiv K_1 = B\,' / [B\rho]$$

$$\kappa(\mathrm{m}^{-2}) = 0.299792[B\,'(\mathrm{T/m})]/[p(\mathrm{GeV})]$$

- **Effective Length**

$$L = [B\,'(0)]^{-1} \int B\,'(z)\, dz \;\sim (0.75 - 0.97) \times (\text{physical length w/coils})$$

- **Equivalent Thin Lens Focal Length**

$$f = 1 / [\kappa L\,] = [B\rho]/[B\,' L\,]$$

$$f\,(\mathrm{m}) = [p(\mathrm{GeV})]/[0.299792\, L(\mathrm{m})\, B\,'(\mathrm{T/m})]$$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

What about higher order optics?

- **Second-order (chromatic aberrations) are (relatively) straightforward**
- **Replace the Reference Momentum with the Actual Momentum (e.g. in the Quadrupole Coefficient K_1) and expand in δ**

$$K_1(p) = [qB'/p] = (p_s/p)\,[qB'/p_s]\,] \equiv (p_s/p)\,K_1(p_s)$$

$$p \equiv (1+\delta)\,p_s$$

$$\text{so:}\quad (p_s/p) \approx 1 - \delta + O(\delta^2)$$

- **Solution to second-order equations of motion found using the Green's function approach for solving differential equations (PBO Lab).**
- **Third-order requires considerably more work**

  - Intrinsic third-order is independent of fringe-fields

  - Fringe-field third-order require four integrals (Matsuda & Wollnik)

- **As part of Task 2:**

  - Added fringe-field third-order terms to TRANSPORT, TURTLE

  - Added Electrostatic Quadrupole to TRANSPORT, TURTLE:
  First-, Second- and Third-order: $K_1(p) = 2qV_0/(a^2\beta p)$ & $\beta = \beta(p_s,\delta)$

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Bend

- **Reference Trajectory follows an arc → curvilinear coordinates used**
- **For idealized Sector Dipole, 5 of the 9 submatrices are non-zero:**

$$R_{xx} = \begin{bmatrix} \cos(hs) & \sin(hs)/h \\ -h\sin(hs) & \cos(hs) \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & s/\gamma_s^2 \\ 0 & 1 \end{bmatrix}$$

$$R_{xz} = \begin{bmatrix} 0 & [1-\cos(hs)]/h \\ 0 & \sin(hs) \end{bmatrix} \qquad R_{zx} = \begin{bmatrix} -\sin(hs) & -[1-\cos(hs)]/h \\ 0 & 0 \end{bmatrix}$$

where $h = 1/\rho$ **and** ρ **is the bend radius**

- **Dispersion & compaction are introduced by the** R_{xz} **and** R_{zx} **submatrices**
- **Non-Idealized Sector Dipoles are More Common**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

Equations of Motion - Magnetic Bend (con't)

- **Non-Idealized Sector Dipole Effects that Impact First-Order Optics:**
  - **Fringe Fields**
  - **Pole Face Rotations**
  - **Pole Shoe Rotations**
- **First Order Fringe Field Effects and Pole Face Rotations are Often Referred to a "Edge Focusing"**
  - **Can be Modeled with Thin Lens at Entrance/Exit**
- **Pole Shoe Rotations Result in Non-Uniform B Field**
  - **First Order Effect is Radial Derivative of B Field**
  - **Often Referred to as a Gradient Bend**
  - **Can be Modeled with a Quadrupole Field added to Dipole**
- **Other Deviations Can also Useful**
  - **Curved Pole Faces, Higher-Order Combined Function Bends, ...**

3. Equations of Motion: Drifts, Quads, Bends …(cont'd)

R-Matrix Example - Mass Separator Bend

- **10 keV Phosphorous (β = 0.000833 and γ = 1.000000)**
- **90° Gradient Bend L of 0.785398 m, Field Index n of 1/2, Pole Tip Field B_o of 0.160 T, bend radius ρ of 50 cm**