Appendix A
Quadrupole Doublets, Triplets & Lattices

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Presentation Outline

Overview of Particle Beam Optics
Appendix A: Doublets, Triplets and Lattices

A.1 Quadrupole Doublet

A.2 Quadrupole Triplet

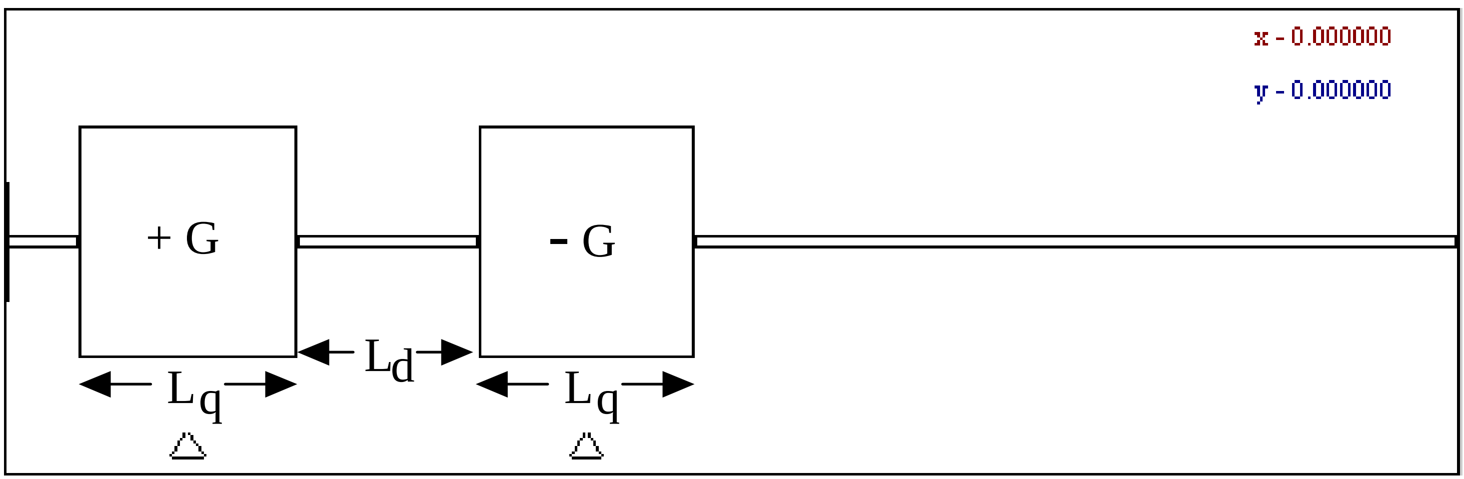
A.3 FODO Lattice

A.4 Summary

A.1. Quadrupole Doublet

- **Two opposite polarity quadrupoles separated by a drift distance** L_d
- **Antisymmetric doublet:**
  - Quadrupole gradients (G_i**) equal in magnitude:** $G_1 = G$ **and** $G_1 = -G$
  - Quad lengths equal: $L_1 = L_2 = L_q$
  - Focusing in both planes possible, for certain values of G**,** L_q**,** L_d

Antisymmetric Quadrupole Doublet



- **Non-Antisymmetric Doublets Offer More Flexibility**

Antisymmetric Quadrupole Doublet

- **Thin lens approximation, the** R-matrix **for the antisymmetric doublet:**
  - **For a quadrupole**: $f_y = -f_x$

$$R_{xx} = \begin{bmatrix} 1 & 0 \\ -1/f_x & 1 \end{bmatrix} \begin{bmatrix} 1 & L_d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ +1/f_x & 1 \end{bmatrix} = \begin{bmatrix} 1+L_d/f_x & L_d \\ -L_d/f_x^2 & 1-L_d/f_x \end{bmatrix}$$

$$R_{yy} = \begin{bmatrix} 1 & 0 \\ +1/f_x & 1 \end{bmatrix} \begin{bmatrix} 1 & L_d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_x & 1 \end{bmatrix} = \begin{bmatrix} 1-L_d/f_x & L_d \\ -L_d/f_x^2 & 1+L_d/f_x \end{bmatrix}$$

- **Thin lens approximation: both planes always focusing** $f = f_x^2 / L_d > 0$
  magnifications in two planes different

Antisymmetric Quadrupole Doublet

- **Thick lens results**

$$R_{xx} = \begin{bmatrix} \cos(kL_q) & \sin(kL_q)/k \\ -k\sin(kL_q) & \cos(kL_q) \end{bmatrix} \begin{bmatrix} 1 & L_d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh(kL_q) & \sinh(kL_q)/k \\ k\sinh(kL_q) & \cosh(kL_q) \end{bmatrix} = \begin{bmatrix} M_x & L_{eff} \\ -1/f & M_y \end{bmatrix}$$

$$R_{yy} = \begin{bmatrix} \cosh(kL_q) & \sinh(kL_q)/k \\ k\sinh(kL_q) & \cosh(kL_q) \end{bmatrix} \begin{bmatrix} 1 & L_d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(kL_q) & \sin(kL_q)/k \\ -k\sin(kL_q) & \cos(kL_q) \end{bmatrix} = \begin{bmatrix} M_y & L_{eff} \\ -1/f & M_x \end{bmatrix}$$

- **Where the four elements** $1/f$, M_x, M_y, L_{eff} **are given by:**

$$1/f = k[\sin(kL_q)\cosh(kL_q) - \cos(kL_q)\sinh(kL_q) + (kL_d)\sin(kL_q)\sinh(kL_q)]$$

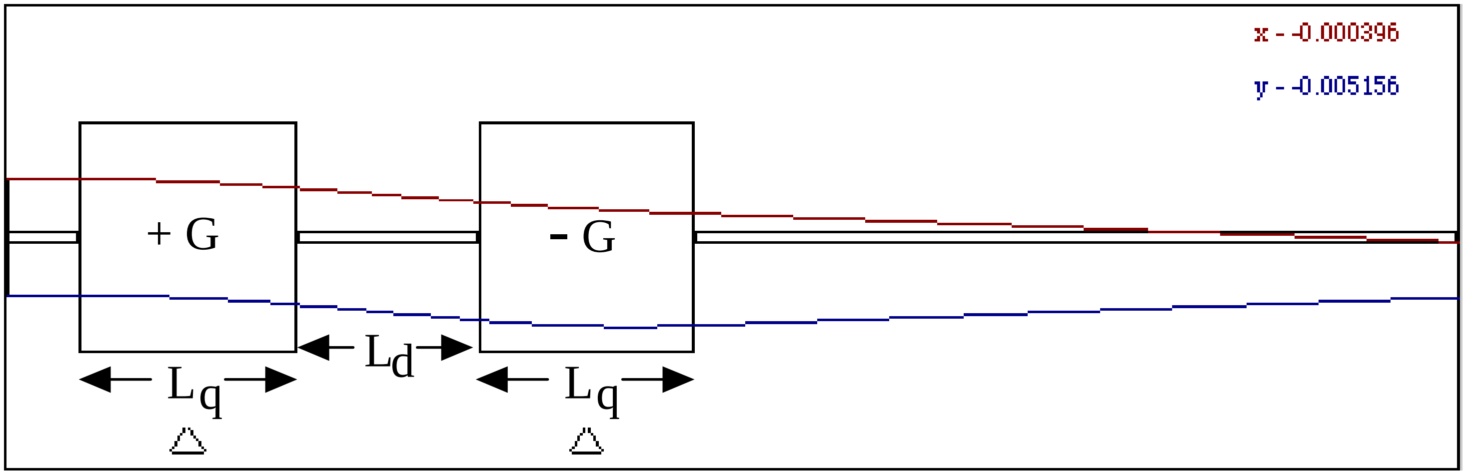
$$L_{eff} = [\sin(kL_q)\cosh(kL_q) + \cos(kL_q)\sinh(kL_q) + (kL_d)\cos(kL_q)\cosh(kL_q)]/k$$

$$M_x = [\cos(kL_q)\cosh(kL_q) + \sin(kL_q)\sinh(kL_q) + (kL_d)\cos(kL_q)\sinh(kL_q)]$$

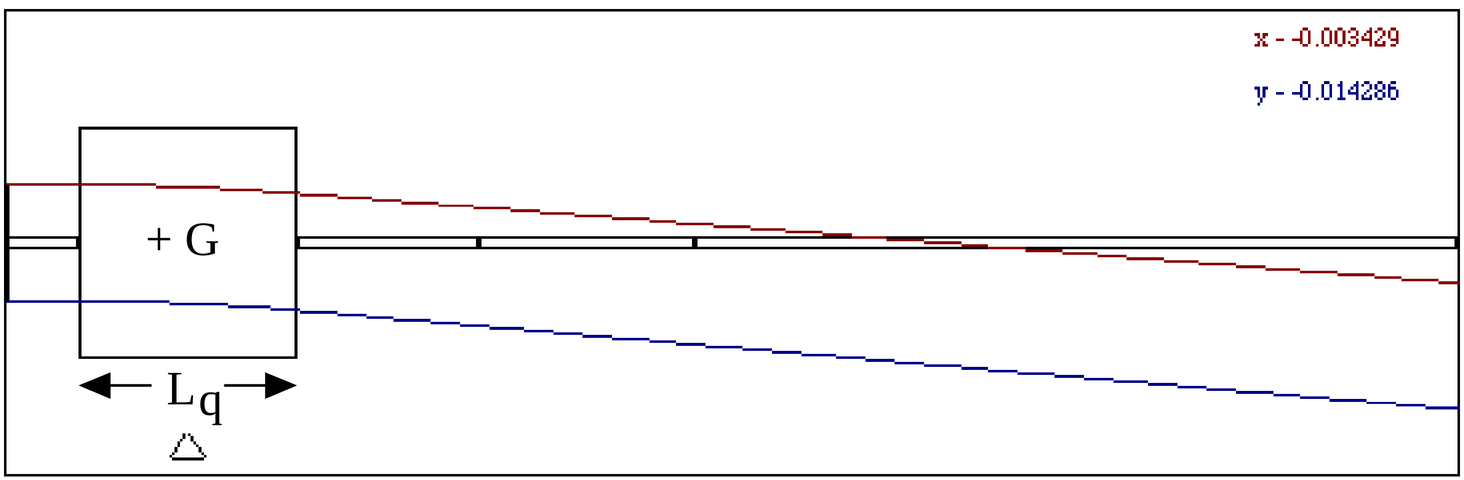
$$M_y = [\cos(kL_q)\cosh(kL_q) - \sin(kL_q)\sinh(kL_q) - (kL_d)\sin(kL_q)\cosh(kL_q)]$$

- **Apparently the effective focal length** f **can have either sign:**
Example 1: @ $kL_q = \pi/2$, $1/f = \pi[\cosh(\pi/2) + (\pi/2)(L_d/L_q)\sinh(\pi/2)]/(2L_q) > 0$
Example 2: @ $kL_q = \pi$, $1/f = -\pi[\sinh(\pi)]/L_q < 0$

Antisymmetric Quadrupole Doublet



Doublet & Singlet for same initial ray: $(q_i) = (0.005,0,0.005,0,0,0)^T$



Antisymmetric Quadrupole Doublet

- **Condition for positive focal length (**$f>0$**)**

  - Equivalently $R_{12} = -1/f < 0$ **or**

  $R_{21} = -k[\sin(kL_q)\cosh(kL_q) - \cos(kL_q)\sinh(kL_q) + (kL_d)\sin(kL_q)\sinh(kL_q)] < 0$

  - Since $k = |K_1|^{1/2} > 0$ **this condition is**

  $- [\sin(kL_q)\cosh(kL_q) - \cos(kL_q)\sinh(kL_q) + (kL_d)\sin(kL_q)\sinh(kL_q)] < 0$ **or**

  $- (kL_d)\sin(kL_q)\tanh(kL_q) < [\sin(kL_q) - \cos(kL_q)\tanh(kL_q)]$

  - Evidently interested in the case where $0 < kL_q < \pi$

  $\sin(kL_q) > 0$ **so**

  $(kL_d) > [\cos(kL_q)/\sin(kL_q)] - [\cosh(kL_q)/\sinh(kL_q)] = [\mathrm{cotan}(kL_q) - \mathrm{cotanh}(kL_q)]$

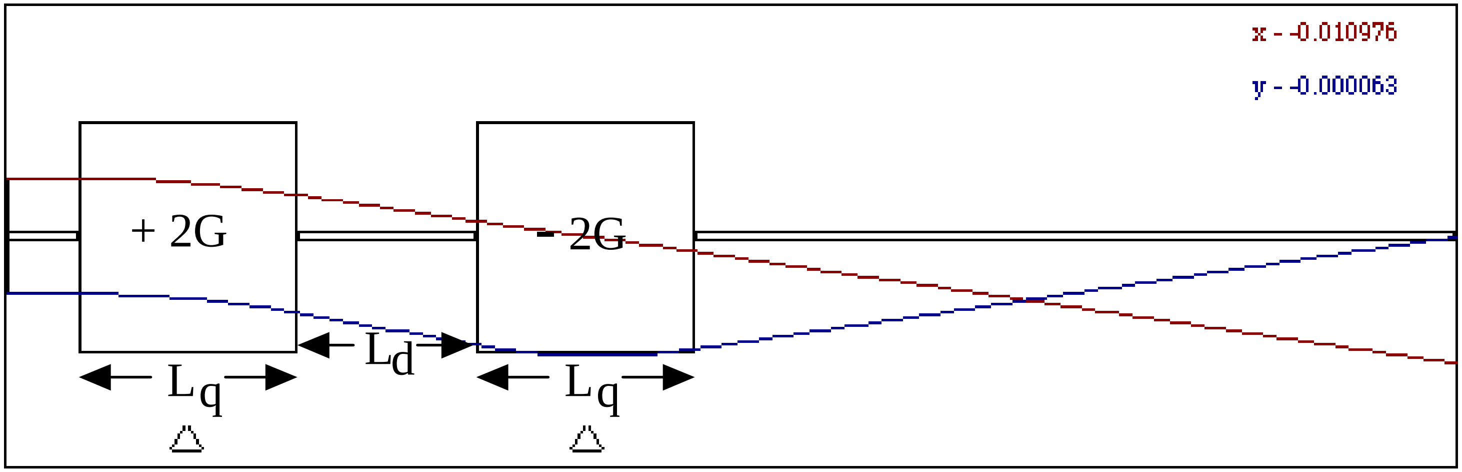
- **Effective focal length** f **positive for "small"** kL_q**, or in terms of** R_{21}**:**

  $$R_{21} = -k[(kL_q)^3(1 + L_d/L_q)] < 0$$

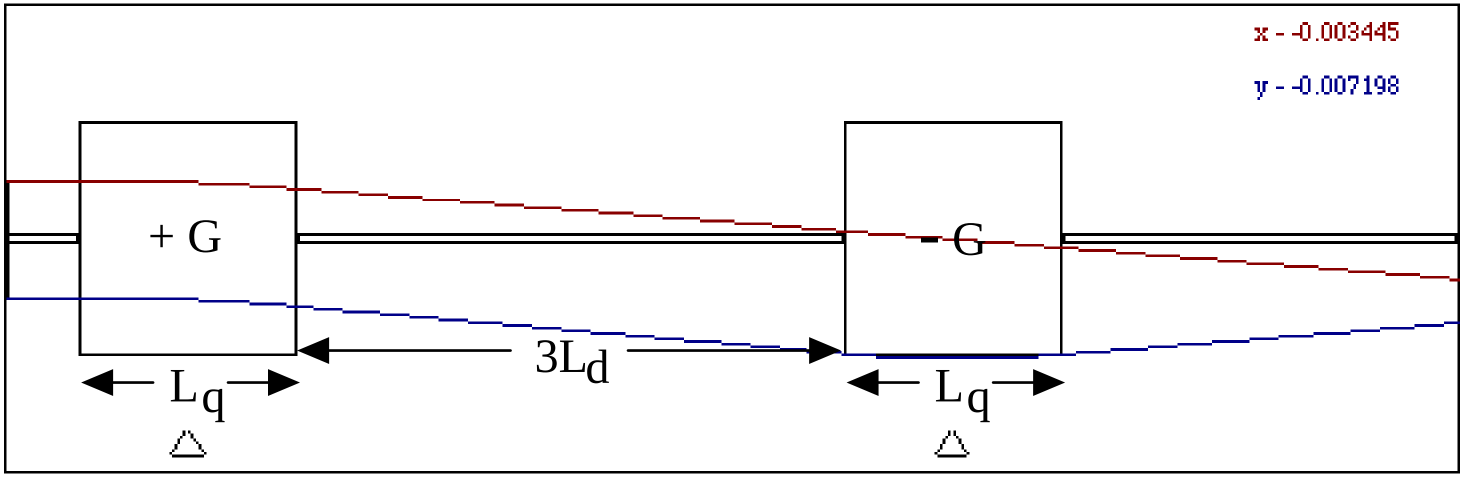
- **Thin lens limit (**$L_d >> L_q$**):**

  $$R_{21} = -k[(kL_q)^3(L_d/L_q)] = -(k^2L_q)^2\, L_d = -L_d/f_x^2$$

Antisymmetric Quadrupole Doublet



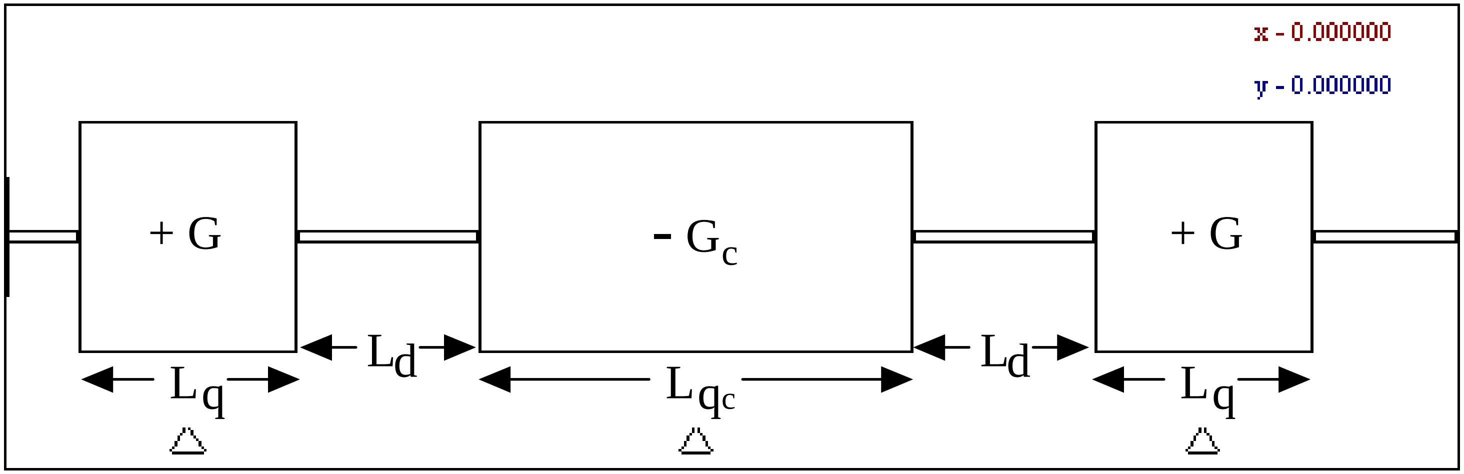
Two Doublets for same initial ray: $(q_i) = (0.005, 0, 0.005, 0, 0, 0)^T$



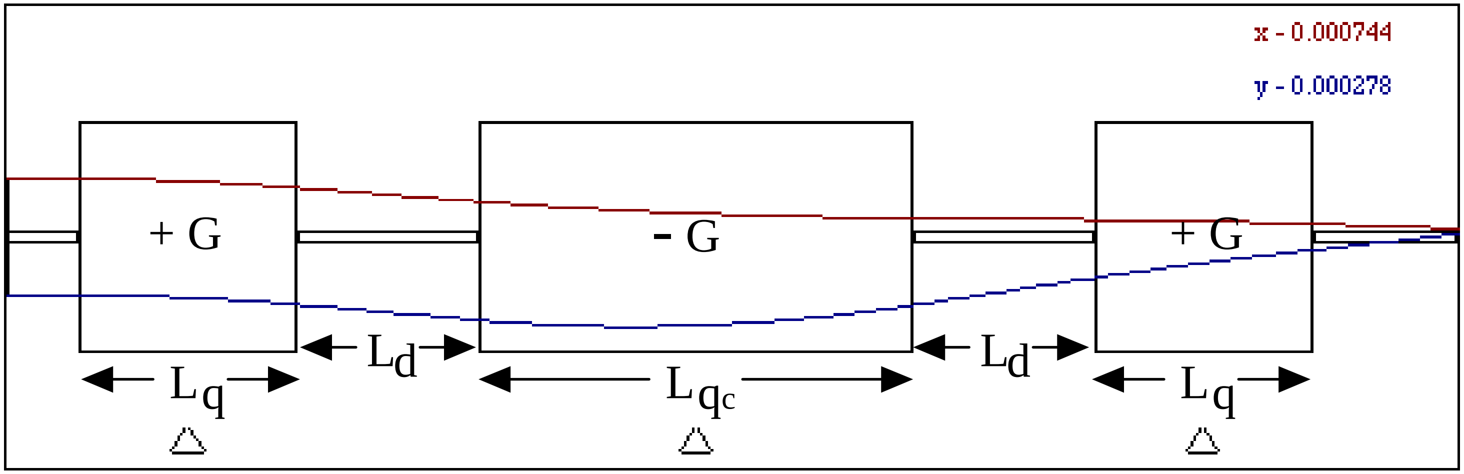
A.2 Quadrupole Triplet

- **Three quadrupoles separated by a drift distances, 2 outer quads have same polarity, inner quad opposite polarity to outer quads**
- **Symmetric triplet:**
  - Outer quad gradients (G_i) equal: $G_1 = G$ **and** $G_3 = G$
  - Outer quad lengths (L_i) also equal: $L_1 = L_3 = L_q$
  - Two plane focusing possible, for certain values of G, L_q, $-G_c$, L_{qc}, L_d
  - Location of principal planes nearly independent of strength

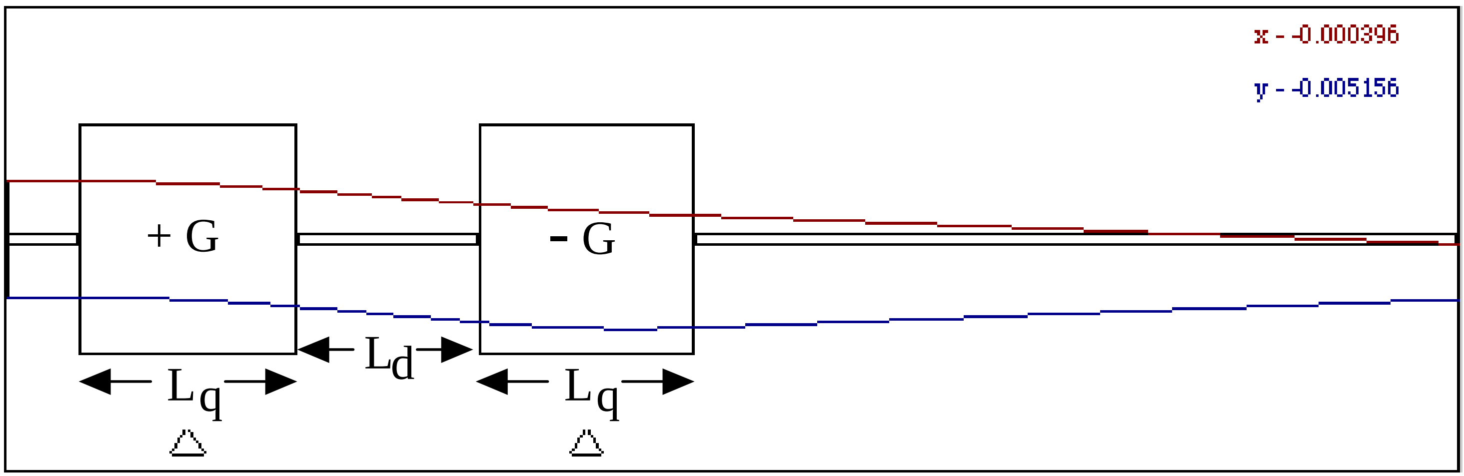
Symmetric Quadrupole Triplet



Example Symmetric Quadrupole Triplet

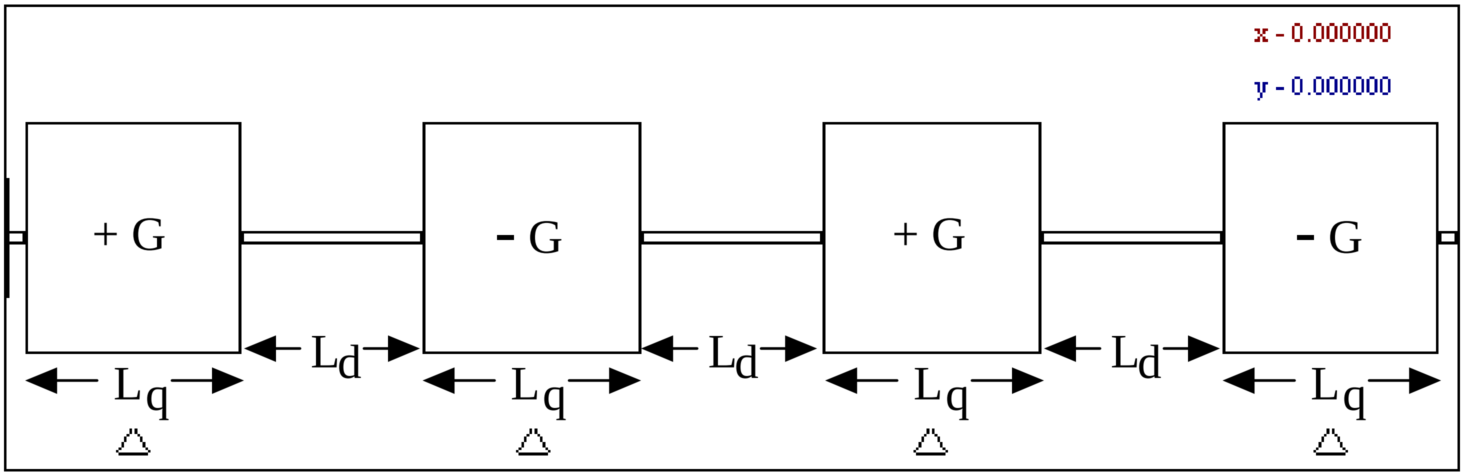


Triplet & Doublet for same initial ray: $(q_i) = (0.005,0,0.005,0,0,0)^T$



A.3. FODO Lattice

- **Periodic array of Antisymmetric Doublets with equal spacing**



- **Construct FODO lattice out of cells, each cell comprised of an Antisymmetric Doublet with two adjacent Drifts of length** $L_d/2$

One Cell of a FODO Lattice

- **Thin lens approximation, the** R-matrix **for the one FODO cell:**

$$R_{yy} = \begin{bmatrix} 1 & L_d/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-L_d/f_x & L_d \\ -L_d/f_x^2 & 1+L_d/f_x \end{bmatrix} \begin{bmatrix} 1 & L_d/2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-L_d/f_x - L_d^2/(2f_x^2) & 2L_d - L_d^3/(2f_x^2) \\ -L_d/f_x^2 & 1+L_d/f_x - L_d^2/(2f_x^2) \end{bmatrix}$$

$$R_{xx} = \begin{bmatrix} 1 & L_d/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+L_d/f_x & L_d \\ -L_d/f_x^2 & 1-L_d/f_x \end{bmatrix} \begin{bmatrix} 1 & L_d/2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+L_d/f_x - L_d^2/(2f_x^2) & 2L_d - L_d^3/(2f_x^2) \\ -L_d/f_x^2 & 1-L_d/f_x - L_d^2/(2f_x^2) \end{bmatrix}$$

- **Focusing in both planes, but is the motion stable over repeated cells?**
  - **- Trace of submatrices:** $\mathrm{Tr}[R_{xx}] = \mathrm{Tr}[R_{yy}] = 2[1-L_d^2/(2f_x^2)]$
  - **- Stability condition (**$|(1/2)\mathrm{Tr}[R]| \leq 1$**):** $|[1-L_d^2/(2f_x^2)]| \leq 1 \Rightarrow f_x \geq L_d/2$

One Cell of a FODO Lattice

- **With thick quadrupole lenses, the** R-matrix **for the one FODO cell:**

$$R_{xx} = \begin{bmatrix} 1 & L_d/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_x & L_{eff} \\ -1/f & M_y \end{bmatrix} \begin{bmatrix} 1 & L_d/2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} M_x - L_d/(2f) & L_{eff} - L_d^2/(4f) + (M_x + M_y)L_d/2 \\ -1/f & M_y - L_d/(2f) \end{bmatrix}$$

R_{yy} **has the same form, but with the interchange of** M_x M_y

- **Stability condition (** $|(1/2)\mathrm{Tr}[R]| \leq 1$ **) yields:**

$$|(1/2)\mathrm{Tr}[R]| = (1/2)\left| M_x + M_y + (L_d/f) \right|$$

$$= \left| \cos(kL_q)\cosh(kL_q) + (kL_d)[\cos(kL_q)\sinh(kL_q) - \sin(kL_q)\sinh(kL_q)] + (1/2)(kL_d)^2\sin(kL_q)\sinh(kL_q) \right| < 1$$

Stability Condition for Another Type of "Lattice"

- **Consider four 90° S-Bends, separated by drifts of length** ds
- **The** R_{xx} **submatrix for each (idealized) S-Bend becomes:**

$$R_{xx} = \begin{bmatrix} \cos(hs) & \sin(hs)/h \\ -h\sin(hs) & \cos(hs) \end{bmatrix} = \begin{bmatrix} 0 & \rho \\ -1/\rho & 0 \end{bmatrix}$$

- **Construct a cell of one S-Bend with drifts of** ds/2 **on each side:**

$$R_{xx} = \begin{bmatrix} 1 & ds/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \rho \\ -1/\rho & 0 \end{bmatrix} \begin{bmatrix} 1 & ds/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -ds/(2\rho) & \rho - L^2/(4\rho) \\ -1/\rho & -ds/(2\rho) \end{bmatrix}$$

- **Stability condition (**$|(1/2)\mathrm{Tr}[R]| \leq 1$**) yields:**

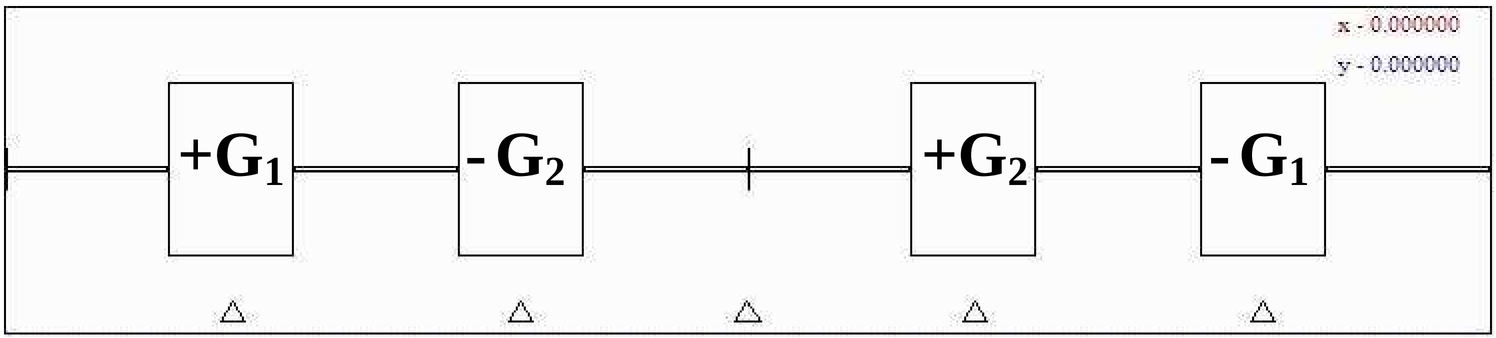
$$|(1/2)\mathrm{Tr}[R]| = (1/2)\left| -ds/(2\rho) - ds/(2\rho) \right| = ds/(2\rho) < 1$$

- **Stability condition for a simple ring of 4 90° S-Bends with Reference Trajectory radius** ρ**, each separated by distance** ds**:**

$$ds < 2\rho$$

"Russian" Quadruplet

- **Variant of a FODO cell or**
- **Two non-antisymmetric doublets with a "FODO" polarity**



- **Apparently offers some advantages in reducing aberrations**

A.4. Summary

Overview of Particle Beam Optics
Appendix A: Doublets, Triplets and Lattices

A.1	**Quadrupole Doublet**	Useful building block
A.2	**Quadrupole Triplet**	
A.3	**FODO Lattice, simple synchrotron lattice**	Stability condition